

Coding Theory, Cryptography, and Finite Fields

Technology and Applications

Edited by
Anand Nayyar, R. S. Raja Durai, Shalini Gupta, Neetu Dhiman and Ashwini Kumar

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Coding Theory, Cryptography, and Finite Fields

Coding Theory, Cryptography, and Finitie Fields explores the theory and applications of finite fields, from basic concepts such as modular arithmetic and polynomial interpolation to advanced topics such as the construction and counting of polynomials. This book:

- Addresses a wide range of topics within coding theory, cryptography, and finite fields, offering a holistic understanding of these interconnected fields.
- Discusses cutting-edge subjects such as quantum cryptography and multimedia security, making it relevant to current and future technological advancements.
- Explores polynomials over finite fields and their applications in coding theory and cryptography to enhance the reader's understanding of the mathematical foundations critical to these areas.
- Emphasizes real-world applications of theoretical concepts, particularly in error correction, data security, and encryption algorithms.
- Explains critical challenges in protecting multimedia data, offering cutting-edge techniques for securing images, videos, and audio files against unauthorized access and piracy.

The text is primarily written for senior undergraduates, graduate students, and academic researchers in electrical engineering, electronics and communications engineering, computer science and engineering, and information technology.


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Preface

The interplay between *coding theory*, *cryptography*, and *finite fields* has long stood at the heart of secure and efficient information processing. From the error-correcting codes that make deep-space communication possible to the cryptographic algorithms that protect our digital identities and financial systems, these mathematical disciplines continue to shape the backbone of modern technology.

The edited book titled *Coding Theory, Cryptography, and Finite Fields: Technology and Applications* presents a comprehensive and up-to-date survey of recent advancements in these intertwined areas. The book comprises 14 chapters contributed by leading researchers and experts from around the world, each offering valuable insights into contemporary theoretical breakthroughs, emerging technologies, and real-world applications.

Chapter 1, "**An Introduction to Single and Double Error Correcting Codes**," provides a foundational introduction to single and double error-correcting codes. It lays out the core algebraic structures and decoding mechanisms that underpin modern communication systems. **Chapter 2**, "**Skew Generalized Polycyclic Codes with Derivations**," delves deeper into algebraic constructions with a focus on skew generalized polycyclic codes using derivations, offering fresh perspectives on the design of flexible and efficient error-correcting codes. **Chapter 3**, "**Construction of Block-Matrix LCD Codes**," discusses innovative techniques for constructing block-matrix linear complementary dual (LCD) codes, known for their robustness against side-channel attacks. **Chapter 4**, **"Challenges in Multimedia Data,"** provides an extensive survey of the challenges involved in protecting multimedia data, emphasizing the intricacies of securing audio, video, and image content through encryption and watermarking. **Chapter 5**, "**Hybrid Cryptographic Systems: The Need for Security, Cryptography and Advancements**," provides a detailed analysis of hybrid cryptographic systems, highlighting how the synergy of symmetric and asymmetric encryption can bolster security in diverse applications. In **Chapter 6**, "**Secure Image Encryption Based on a Novel 2D Chaotic Map and Four-Phase Scrambling**," a novel image encryption algorithm is proposed, based on a newly formulated two-dimensional chaotic map and a four-phase scrambling technique aimed at enhancing resistance to cryptanalytic attacks. **Chapter 7**, "**Quantum Key Distribution Protocols**," reviews key protocols in quantum key distribution (QKD). **Chapter 8**, "**Exploring the Evolution of Quantum Teleportation Research in Quantum Cryptography: A Scientometric Evaluation**," enlightens a scientometric evaluation of the research landscape surrounding quantum teleportation in cryptography. **Chapter 9**, "**Quantum Phase Parameterized True Random Number Generation for Secure Communications**," introduces a quantum phase-based true random number generator, demonstrating its effectiveness through statistical analysis. Addressing the post-quantum cryptographic shift, **Chapter 10**, "**From Observations to Models: Statistical Methods in Cosmology**," bridges mathematical techniques with cosmology, showcasing the application of statistical and finite-field methods in modeling cosmic phenomena and analyzing observational data. **Chapter 11**, "**Progress in Lattice-Based Digital**

Signature: Systematic Review, Challenges, and Research Directions," systematically reviews the current progress and open problems in lattice-based digital signatures. The mathematical core of the volume is further enriched in **Chapter 12**, "**Construction of Some New Classes of Irreducible and Normal Polynomials over Finite Fields**," which constructs new families of irreducible and normal polynomials over finite fields, supported by rigorous proofs of their algebraic properties. **Chapter 13**, "**Enumeration of Latin Squares Constructed Using Bivariate Permutation Polynomials over Finite Fields**," explores Latin squares via bivariate permutation polynomials, offering insights valuable to both combinatorics and coding theory. In applied cryptography, **Chapter 14**, "**PQLS-EVote: A Novel Post-Quantum Secure and Efficient Lattice-Based Signcryption Scheme for Electronic Voting**," presents a lattice-based signcryption scheme tailored for secure electronic voting, combining post-quantum security with practical efficiency.

We extend our sincere gratitude to all the contributing authors for their scholarly work, to the reviewers for their critical and constructive feedback, and to the publishing team for their unwavering support. We also acknowledge the collaborative spirit of the mathematical and cryptographic community that continues to drive innovation and discovery.

As editors, we hope this volume will not only serve as a useful resource for current developments but also inspire future research and innovation in the rich and ever-evolving domains of coding theory, cryptography, and finite fields.

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Anand Nayyar received a Ph.D. in Computer Science from Desh Bhagat University, Mandi Gobindgarh, India in 2017 in the area of Wireless Sensor Networks, Swarm Intelligence, and Network Simulation. He is currently working in the School of Computer Science and Artificial Intelligence (SCA), Duy Tan University, Da Nang, Vietnam, as Professor, Scientist, Vice-Chairman (Research), and Director of IoT and Intelligent Systems Lab. He is a Certified Professional with 275+ professional certifications from CISCO, Microsoft, CompTIA, Amazon, Alibaba Cloud, Oracle, Google, Tableau, FinOps, Beingcert, EXIN, GAQM, Cyberoam, and many more. He has published more than 200+ research papers in various high-quality ISI-SCI/SCIE/SSCI Impact Factor—Q1, Q2, Q3, and Q4 journals and Scopus/ESCI-indexed journals, 80+ papers in international conferences indexed with Springer, IEEE, and ACM Digital Library; 60+ book chapters in various Scopus/Web of Science-indexed books with various reputed journals with citations: Google Scholar: 23,000+, H-Index: 78 and I-Index: 320; Scopus: 13,000+; H-index: 60. He has been a member of more than 60+ associations as Senior and Life Member like IEEE (Senior Member) and ACM (Senior Member). He has authored/co-authored and edited 80+ books on Computer Science. He is associated with more than 600+ international conferences as Programme Committee/Chair/Advisory Board/Review Board member. He has completed 1 grassroots and 1 ASEAN project. He has 18 Australian patents, 16 German patents, 4 Japanese patents, 43 Indian Design cum Utility patents, 13 UK patents, 1 US patent, 3 Indian copyrights, and 2 Canadian copyrights to his credit in the area of Wireless Communications, Artificial Intelligence, Cloud Computing, IoT, Healthcare, Drones, Robotics, and Image Processing. He has been presented 60 awards for Teaching and Research—Young Scientist; Best Scientist; Best Senior Scientist; Asia Top 50 Academicians and Researchers; Young Researcher Award; Outstanding Researcher Award; Excellence in Teaching; Best Senior Scientist Award; DTU Best Professor and Researcher Award—2019, 2020–2021, 2022, 2022–2023, 2023–2024, 2024–2025; Distinguished Scientist Award by National University of Singapore, Obada Prize 2023, Lifetime Achievement Award 2023, 2024; Asian Admirable Achievers 2024; Distinguished Academic Leader 2024, Lifetime Achievement Award 2024; and many more.

He is listed in Top 2% Scientists as per Stanford University (2019, 2020, 2021, 2022, 2024, 2025), Ad Index (Rank No. 1 Duy Tan University, Rank No. 2 Computer Science in Vietnam) and listed on *Research.com* (Top Scientist of Computer Science in Vietnam—National Ranking: 2; D-Index: 56; World Ranking: 3694).

He has been acting as Associate Editor for several reputed journals. He is acting as Managing Editor for *IJKSS (IGI-Global) Journal*. He has reviewed more than 5,500+ articles for diverse Web of Science and Scopus-indexed journals. He is currently researching in the area of Wireless Sensor Networks, Internet of Things, Swarm Intelligence, Cloud Computing, Artificial Intelligence, Drones, Blockchain, Cyber Security, Healthcare Informatics, Big Data, and Wireless Communications.

R. S. Raja Durai is currently Professor in the Mathematics Department of JUIT (January 2023–Present). He served as Associate Professor in the Mathematics Department of JUIT (2013–December 2022). He joined the institute as Assistant Professor in 2008. He received his bachelor's degree (Loyola College) and master's degree (IIT Madras) in Mathematics in 1997 and 1999, respectively. In 2004, he completed his doctoral research in the field of *Algebraic Coding Theory* from the Department of Mathematics, Indian Institute of Technology Madras. He then carried out his first post-doctoral research on *sequence design for mobile and wireless communications* under the supervision of Prof. Naoki Suehiro at the University of Tsukuba, Japan. His second post-doctoral project was a research-cum-implementation work on *distributed video coding* with IRISA/INRIA, France. He is primarily interested in the research and implementation of his research work in real channels.

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Ashwini Kumar received his bachelor's and master's degrees in Mathematics in 2012 and 2015, respectively. He received a Master of Philosophy in Applied Mathematics in 2016 from Himachal Pradesh University. He completed his doctoral research in Algebraic Coding Theory in 2023, under the supervision of Dr. R. S. Raja Durai at the Department of Mathematics, Jaypee University of Information Technology, Solan, Himachal Pradesh, India. Currently, he is working as Assistant Professor at the Department of Applied Sciences, Chitkara University, Punjab, India.

His research focuses on developing error-control coding schemes for reliable and efficient data transmission in mobile and wireless networks, bridging the gap between theory and practice. He is driven by a passion for making significant contributions to the field of information theory and communications and is dedicated to collaborating with international experts to advance the field of communications technology.

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Chapter 1

An Introduction to Single- and Double-Error-Correcting Codes

T. J. Cinderella

1.1 INTRODUCTION

Error-correcting codes were initially developed to address the problem of errors introduced during data transmission over noisy and unreliable communication channels. The theoretical foundation of this field was laid by Claude Shannon (1948) through his groundbreaking work on the concept of channel capacity and the limits of reliable communication over noisy channels. Building on this, Richard Hamming (1950) introduced the first practical error-correcting code—the (7,4) Hamming code—which is capable of detecting and correcting single-bit errors, marking the beginning of practical error correction.

However, in real-world scenarios, it is common for errors to affect more than one bit. To address this, more advanced codes were developed to correct multiple-bit errors. Notably, during the 1950s and 1960s, several powerful error-correcting codes were introduced, including the Bose–Chaudhuri–Hocquenghem (BCH) codes (Bose & Ray-Chaudhuri, 1960; Hocquenghem, 1959), Reed–Muller codes, and Reed–Solomon codes (Hill, 1986; Blahut, 2003), which significantly improved the reliability of digital communication systems.

In a typical communication model involving a binary symmetric channel (BSC), the sender encodes the message by appending redundant bits to the original information bits, forming a code word (Cover & Thomas, 2006; Yeung, 2008; Lin & Lee, 2004). This code word is transmitted through the channel, which may introduce random errors due to noise. The receiver then applies a decoding algorithm to reconstruct the original message (Peterson & Weldon, 1972; Roman, 1992; Pless, 1998).

One widely used decoding technique is based on the minimum Hamming distance criterion. The decoder compares the received word with all valid codewords and selects the one with the smallest Hamming distance—this is known as maximum likelihood decoding (Blahut, 1983; MacWilliams & Sloane, 1977). If a unique code word satisfies this condition, it is chosen as the decoded message. Otherwise, ambiguity arises, and the decoder may request retransmission or apply additional error detection techniques (Berlekamp, 1968/2015; Huffman & Pless, 2003).

The minimum distance d of a code determines its error-correcting capability. Codes with $d = 3$ can correct single-bit errors, while those with $d = 5$ can correct up to two errors (Hill, 1986; Vanstone & van Oorschot, 1989; Blahut, 1983). These principles are foundational in the study of linear block codes, cyclic codes, and BCH codes, which are explored in several standard texts (MacWilliams & Sloane, 1977; Roman, 1992; Calderbank & Sloane, 1996).

In practice, single-error-correcting and double-error-detecting codes are widely used in hardware-level error control, particularly in memory systems and communication protocols (Lin & Costello, 2004; Peterson & Weldon, 1972; Berlekamp, 1963). More sophisticated algorithms like syndrome decoding, the Viterbi algorithm (Forney, 1973), and algebraic decoding techniques (Sudan, 1997; Berlekamp, 1968) had been developed to handle higher error rates.

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This chapter introduces the fundamentals of single- and double-error-correcting codes and lays the groundwork for further exploration of more advanced error-control coding techniques, such as convolutional codes and low-density parity check (LDPC) codes (Beth et al., 1999; Blake, 1973; Justesen, 1972; Kasami, 1966; Solomon & Stiffler, 1964).

Key references that guide the study of error-control coding include classic texts by Mac Williams and Sloane (1977), Pless (1998), Lin and Costello (2004), and Huffman and Pless (2003), as well as foundational papers and journal articles by Elias (1954), Hamming (1950), Shannon (1948), Forney (1973), and others who shaped modern coding theory.

The main objectives of this chapter include introducing the fundamental principles of linear error-correcting codes with a focus on the Hamming (7,4) code and its decoding techniques, followed by the concepts of dual codes, code word weight, and syndrome decoding in the context of error detection and correction. In addition, we focus on the construction and properties of maximum distance separable (MDS) codes and BCH codes.

Organization of the Chapter

The rest of the chapter is organized as: Section 1.2 presents the concept of maximum likelihood decoding, Section 1.3 introduces syndrome decoding, Section 1.4 discusses the packing radius and covering radius of codes, providing geometric insights into code performance. Section 1.5 examines MDS codes and their error-correcting capabilities. Section 1.6 details the construction of finite fields, particularly the field with 16 elements, which is crucial for advanced code construction. Section 1.7 focuses on BCH codes, outlining their construction and multierror correction techniques. And, finally, section 1.8 and 1.9 concludes the chapter with future scope.

In the theory of error-correcting codes, we discuss methods to construct efficient codes, which enable us to detect and correct errors. One such efficient code is the Hamming code. Hamming codes consist of 16 code words, which are obtained by taking all possible combinations of the rows of the following matrix.

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Some of the code words in the Hamming code are the following:

$$\begin{array}{l} (0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0) \\ (1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1) \\ (0\ \ 1\ \ 0\ \ 0\ \ 1\ \ 0\ \ 1) \\ (0\ \ 0\ \ 1\ \ 0\ \ 1\ \ 1\ \ 0) \\ (0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 1\ \ 1) \end{array}$$

Hamming codes can detect and correct single errors. Consider the following vectors:

$$\begin{aligned} \vec{a} &= (0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 1\ \ 1) \\ \vec{b} &= (0\ \ 1\ \ 1\ \ 0\ \ 0\ \ 1\ \ 1) \\ \vec{c} &= (1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1) \end{aligned}$$

Let us see how the error-correcting operation is done in the Hamming code. Consider a code word, $x = (0\ \ 1\ \ 1\ \ 0\ \ 0\ \ 1\ \ 1)$. Suppose x is sent and received as $y = (0\ \ 1\ \ 1\ \ 0\ \ 0\ \ 1\ \ 1)$. Then, we compute $y.\vec{a}, y.\vec{b}$, $y.\vec{c}$. Here $y.\vec{a} = 0, y.\vec{b} = 0, y.\vec{c} = 0$. The number formed by $y.\vec{a}, y.\vec{b}$

is $(0\ \ 0\ \ 0)$, which implies that there is no error in the received vector. Next consider $x=(1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1)$. Suppose the received vector is $y=(1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 1)$. Computing $y.\vec{a}, y.\vec{b}, y.\vec{c}$, we get $y.\vec{a}=1, y.\vec{b}=1,\ y.\vec{c}=0$. The error corresponding to $y.\vec{a}, y.\vec{b}, y.\vec{c}$ is $(1\ \ 1\ \ 0)$, indicating there is a single error in the sixth position.

Definition 1.1.1 *A linear (n,k) code is a k-dim subspace of an n-dim vector space V.*

Hamming code is a (7,4) code, being a four-dimensional subspace of the vector space of all binary seven-tuples.

Remark 1.1.2 *A finite field of order q is usually referred to as a Galois field of order q and is denoted by $GF(q)=\{0,1,\ldots,q-1\}$. For binary codes, the field is supposed to be GF(2), for ternary codes it is GF(3), and so on.*

Another set of important codes is represented by American Standard Code for Information Interchange (ASCII). This is an overall parity check code. In ASCII code, the first seven symbols are information symbols, and one bit is added to make the number of 1's even in each code word. This bit is referred to as the parity check bit. ASCII codes can detect a single error but cannot correct it.

There are two ways to introduce a code: one is using a generator matrix, and the other is using a parity check matrix. A generator matrix consists of the basic elements of the code as its rows. By taking all possible linear combinations of the generator matrix, we can construct the code. If G is a generator matrix of an (n,k) code C, then any set of k columns of G that are independent is called an information set.

Hamming (7,4) codes are given by the generator matrix,

$$H=\begin{bmatrix}1&0&0&0&0&1&1\\0&1&0&0&1&0&1\\0&0&1&0&1&1&0\\0&0&0&1&1&1&1\end{bmatrix}$$

The list of all code words of the binary Hamming code is given by:

(0 0 0 0 0 0 0)
(1 0 0 0 0 1 1)
(0 1 0 0 1 0 1)
(0 0 1 0 1 1 0)
(0 0 0 1 1 1 1)
(1 1 0 0 1 1 0)
(1 0 1 0 1 0 1)
(1 0 0 1 1 0 0)
(0 1 1 0 0 1 1)
(0 1 0 1 0 1 0)
(0 0 1 1 0 0 1)
(1 1 1 0 0 0 0)
(1 1 0 1 0 0 1)
(1 0 0 1 0 1 0)
(0 1 1 1 1 0 0)
(1 1 1 1 1 1 1)

In Hamming codes, the first four symbols of every code word are information symbols, and the rest are redundancy symbols. As we have seen, Hamming codes can detect and correct only single errors. Let's see why the redundancy symbols are important:

Consider a BSC with a probability of correct transmission of 0.9. Then the probability of an error is 0.1. Consider the Hamming code again. Suppose we send the four information symbols alone through this channel. Then the probability of receiving it correctly is $(0.9)^4 = 0.6561$. Roughly the probability of receiving a message correctly if only the information symbols are sent is 65%. On the contrary, suppose that we send a fully encoded Hamming code word. The received message can be correctly decoded if there is up to one error only. Therefore, here the probability will be $(0.9)^7 + 7*(0.9)^6 * 0.1 = 0.8503$. That is, the probability of receiving a message correctly by encoding is 85%, which is a considerable improvement. P_{err} is the probability of incorrectly decoding the message received in a particular decoding scheme. For Hamming codes, $P_{err} = 0.1497$. P_{err} is also known as the word error rate.

Another binary code is C_1, a (5,3) code generated by the generator matrix:

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The code words in C_1 are:

(0 0 0 0 0)
(1 0 0 1 1)
(1 0 0 1 1)
(0 1 0 0 1)
(0 0 1 1 1)
(1 1 0 1 0)
(1 0 1 0 0)
(0 1 1 1 0)
(1 1 1 0 1)

In C_1, the first three positions are information positions, and the last two are redundancy positions. A binary vector $(a_1, a_2, a_3, a_4, a_5)$ is an element of C_1 iff

$$a_1 + a_3 + a_4 = 0 \text{ and } a_1 + a_2 + a_3 + a_5 = 0 \tag{1.1}$$

This is true because if a_i stands for the ith column, then $a_4 = a_1 + a_3$ and $a_5 = a_1 + a_2 + a_3$. The above equations are referred to as the *parity check equations*. Equation (1.1) can also be expressed as:

$$1.a_1 + 0.a_2 + 1.a_3 + 1.a_4 + 0.a_5 = 0,$$
$$1.a_1 + 1.a_2 + 1.a_3 + 0.a_4 + 1.a_5 = 0$$

That is, any code word in C_1 is orthogonal to the rows of the matrix H, given by

$$H = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

H is said to be the *parity check matrix* of C_1.

Definition 1.1.3 *Let H be the parity check matrix of a code* C. *Then* x *is an element of* C *iff* $x.H^T = 0$ *(or)* $x.w = 0$ *for every row* w *of* H.

For an (n,k) code C, the order of its generator matrix is $k \times n$ and the order of its parity check matrix is $(n-k)\times n$.

Definition 1.1.4 *Let* C *be an* (n,k) *code. The dual code of* C *or the orthogonal complement of* C *denoted by* $C^{\perp}$ *is the set of all vectors that are orthogonal to every element of* C.

If C is an (n,k) code, then $C^{\perp}$ will be an $(n,n-k)$ code. If H is the parity check matrix of C, then the code constructed with H as its generator matrix will be $C^{\perp}$.

Theorem 1.1.5 *If an* (n,k) *code* C *has a generator matrix* $G = (I, A)$ *in standard form, then* $H = (-A^T, I)$ *is a parity check matrix of* C, *where* A *is an* $k \times n-k$ *matrix,* A^T *is the transpose of* A, *and* I *is an identity matrix of order* $(n-k)$.

$$\textit{Proof}.\ \text{Let } G = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_k \end{bmatrix}, H = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_{n-k} \end{bmatrix}, \text{ and } A = \begin{bmatrix} a_{11} & \cdots & a_{1,n-k} \\ a_{21} & \cdots & a_{2,n-k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{k,n-k} \end{bmatrix}$$

We shall give a computational proof for the theorem. By the above, $g_1 = (1,0,0\ldots0, a_{11}, a_{12},\ldots, a_{1,n-k})$ and $h_1 = (-a_{11}, -a_{21},\ldots,-a_{k1},1,0,0,\ldots,0)$. Then $g_1.h_1 = -a_{11} + a_{11} = 0$. Similarly, we can show that $g_i.h_j = 0, \forall i, j$. That is, each row of G is orthogonal to every row of H, making H a parity check matrix of C.

Problem 1.1.6 *Give a parity check matrix for a* (6,3) *binary code* C_3 *with generator matrix,*

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

The parity check equations are:

$$a_3 = a_2 + a_4$$
$$a_5 = a_1 + a_4$$
$$a_6 = a_1 + a_2$$

The information positions are first, second, and fourth. That is,

$$a_2 + a_3 + a_4 = 0$$
$$a_1 + a_4 + a_5 = 0$$
$$a_1 = a_2 + a_6 = 0$$

i.e.,

$$0.a_1 + 1.a_2 + 1.a_3 + 1.a_4 + 0.a_5 + 0.a_6 = 0$$
$$1.a_1 + 0.a_2 + 0.a_3 + 1.a_4 + 1.a_5 + 0.a_6 = 0$$
$$1.a_1 + 1.a_2 + 0.a_3 + 0.a_4 + 0.a_5 + 1.a_6 = 0$$

$\therefore$ Parity check matrix, $H = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

Problem 1.1.7 *Give the parity check matrix for the Hamming code.*

We have its generator matrix G,

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Then, by the theorem, its parity check matrix is given by:

$$H = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Problem 1.1.8 *Give the parity check matrix of the* (4,2) *ternary code* C_2 *generated by*

$$G_2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Since being a ternary code –(1) = 2 and –(2) = 1, and so, by the theorem, the parity check matrix is given by:

$$H_2 = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}$$

Definition 1.1.9 *Weight of a code word in a code is the number of non-zero components in it.*

Example 1.1.10 *In code* C_2, *every non-zero code word has weight 3.*

Example 1.1.11 *Consider the* $(n,1)$ *code generated by* $H = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}$. *This code consists of only two code words, namely, the all-zero code word and the all-one code word. This code is referred to as the repetition code. The orthogonal complement of this consists of all even-weight vectors of length* n, *which are generated by*

$$E_n = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \\ 1 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

1.2 MAXIMUM LIKELIHOOD DECODING

Consider a code C and suppose that a code word u is sent through a transmission channel and received as vector v. The problem is decoding v correctly to u. In maximum

likelihood decoding we select u from the list of code words so that v has the minimum distance from u.

The method of decoding in this manner is said to be maximum likelihood decoding. This may be explained in another form also. "If v is the received vector, decode v to u so that e is the vector of smallest weight such that $v-u=e$ or $v=u+e$".

Definition 1.2.1 *The weight of a code C is the smallest of all the weights of the non-zero members of C.*

If d is the weight of an (n,k) code, then it is generally referred to as an (n,k,d) code. For example, Hamming code is a (7,4,3) code.

Example 1.2.2 *Weight of the repetition code of length n is n. The weight of its complement is 2. Weight of the codes C_1 and C_2 is 2 and 3, respectively. The weight of the Hamming code is 3.*

Definition 1.2.3 *The weight distribution of a code is a list of the number of code words of each weight.*

Generally, A_i denotes the number of code words of weight i.

Example 1.2.4 *In Hamming code C, it is clear that $A_0=1$ and $A_7=1$. Since all-one vector is a code word and C is a vector space, there cannot be code words with weight 5 or 6. Since, in such cases, there will be code words of weight 2 or 1, which is impossible for a code of weight 3. Therefore, $A_3=A_4=7$.*

Definition 1.2.5 *The distance between two vectors u and v, denoted by $d(u,v)$, is the number of places in which u and v differ.*

Example 1.2.6 *Suppose $u=(1\ 0\ 0\ 0\ 0\ 1\ 1)$, $v=(0\ 1\ 0\ 0\ 1\ 0\ 1)$, and $w=(1\ 2\ 0\ 2\ 0\ 2\ 1)$, then $d(u,v)=4$ and $d(u,w)=3$. In general, we can see that $d(u,v)$ is the weight of $u-v$. For binary codes, $d(u,v)$ may be taken as weight of $u+v$.*

Definition 1.2.7 *A code is a metric space with respect to the distance d defined by $d(u,v)=wt(u-v)$.*

For,

1. $d(u,v)\geq 0$
2. $d(u,v)=0$ iff $u=v$
3. $d(u,v)=d(v,u)$
4. $d(u,v)=wt(u-v)=wt(u-w+w-v)\leq wt(u-w)+wt(w-v)=d(u,w)+d(w,v)$

Since a code is a metric space, we can think of spheres of radius r with center at a code word u, denoted by $S_r(u)$.

Theorem 1.2.8 *If d is the weight of a code C, then C can correct $t=\left[\frac{d-1}{2}\right]$ or fewer errors and conversely (where $[x]$ denotes the greatest integer less than or equal to x).*

Proof. To prove C can correct a maximum of t errors, it is enough to show that $S_t(u)\cap S_t(v)=\emptyset$. If possible, assume that $S_t(u)\cap S_t(v)\neq\emptyset$, then $\exists w\in S_t(u)\cap S_t(v)$.

$$\begin{aligned}
&\Rightarrow w\in S_t(u) \text{ and } w\in S_t(v)\\
&\Rightarrow d(u,w)\leq t \text{ and } d(w,v)\leq t\\
&\Rightarrow d(u,v)\leq d(u,w)+d(w,v)\\
&\qquad\qquad \leq t+t=2t\\
&\qquad\qquad \leq d-1
\end{aligned}$$

This is a contradiction, since the weight of the code is d.

Conversely part is obvious.

Remark 1.2.9 *If u and v are any two binary vectors, then $u * v$ represents the number of* 1's *common to u and v*. We can easily verify that

$$wt(u+v) = wt(u) + wt(v) - 2(u * v)$$

1.3 SYNDROME DECODING

Since a code C is an abelian subgroup of V, we can think about the coset of C in V. Let's recall some known facts. Every coset of C has the same number of elements as C does. Any two cosets of C are either disjoint or identical. Also V is the union of all cosets of C. If C is an (n,k) code over $GF(q)$ then, C has q^{n-k} cosets.

Let C be a binary (n,k) code. Then, C has 2^k elements of length n. We know that a total of 2^n vectors of length n are there. Let x be an element of C. If x is sent through a channel, it may be received as $x+e$, where e is the error vector, which may be zero. We know that if $x \in C$, then $x+e \in C+e$. All the 2^n vectors of length n are distributed among the 2^{n-k} distinct cosets of C. By maximum likelihood decoding, we presume that only minimum error is occurring. That is, if $x \in C$, it is received at the other end as $x+e$, where weight of e is minimum. In other words, in a coset we have to look for an element of minimum weight, which is referred to as the coset leader. If e is the coset leader in a coset (say) $C', y \in C'$ may be decoded to $y-e$.

For example, consider a code C generated by the matrix $G = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$. Then $C = \{(0\ 0\ 0\ 0), (1\ 0\ 1\ 0), (0\ 1\ 1\ 1), (1\ 1\ 0\ 1)\}$. The list of all code words and the corresponding elements in other cosets is referred to as *standard array*. The standard array for the above code C is as follows.

	Coset leaders			
Code words	(0 0 0 0)	(1 0 1 0)	(0 1 1 1)	(1 1 0 1)
Other cosets	(1 0 0 0)	(0 0 1 0)	(1 1 1 1)	(0 1 0 1)
	(0 1 0 0)	(1 1 1 0)	(0 0 1 1)	(1 0 0 1)
	(0 0 0 1)	(1 0 1 1)	(0 1 1 0)	(1 1 0 0)

Scanning through this standard array, we can decode any received vector to the corresponding vector code word. For example, if we receive $(1\ 0\ 0\ 1)$, it can be decoded to $(1\ 1\ 0\ 1)$, and if we receive $(1\ 1\ 1\ 1)$, we can decode this to $(0\ 1\ 1\ 1)$ or $(1\ 1\ 0\ 1)$. Even if this method is an efficient way to correct errors, we cannot store all the vectors when the length of the code is a big number. So we define the following:

Consider a code C with parity check matrix H. Then for any vector y, *the syndrome* of y is denoted and defined as

$$Syn(y) = \begin{bmatrix} y.h_1 \\ y.h_2 \\ \vdots \\ y.h_{n-k} \end{bmatrix} = H.y^T$$

Formally, the working of syndrome decoding is as follows: Consider an (n,k) code C over $GF(q)$. Find a set of coset leaders and compute the q^{n-k} syndromes corresponding to them. Suppose we receive a vector y. If $\text{Syn}(y)=0$, then y is a code word. When we choose coset leaders, we start with vectors of weight 1 after considering the zero vector. Suppose e is a coset leader of weight 1. If $\text{Syn}(y)=\text{Syn}(e)$, we decode y to $y-e$, (say, x). Here we are decoding y to the closest code word. After selecting all possible vectors of weight 1 as coset leaders, we choose vectors of weight 2 as coset leaders. If e is such a vector and $\text{Syn}(y)=\text{Syn}(e)$, then also we decode y to $y-e$, which is the closest code word to y, and we proceed in this way until we locate y in some coset of C. We continue selecting coset leaders until all the q^{n-k} syndromes are exhausted. Thus, it makes clear that syndrome decoding is a maximum likelihood decoding.

Theorem 1.3.1 *Syndrome decoding is a maximum likelihood decoding.*

Example 1.3.2 *Consider the code in the last example. The parity check matrix H for this code is* $\begin{bmatrix}1&1&1&0\\0&1&0&1\end{bmatrix}$. *Here the syndrome of each coset leader is given by*

$$Syn(0\ \ 0\ \ 0\ \ 0)=\begin{bmatrix}0\\0\end{bmatrix}$$

$$Syn(1\ \ 0\ \ 0\ \ 0)=\begin{bmatrix}1\\0\end{bmatrix}$$

$$Syn(0\ \ 1\ \ 0\ \ 0)=\begin{bmatrix}1\\1\end{bmatrix}$$

$$Syn(0\ \ 0\ \ 0\ \ 1)=\begin{bmatrix}0\\1\end{bmatrix}$$

Theorem 1.3.3 *Each vector in a fixed coset has the same syndrome. Vectors in different cosets possess different syndromes. All possible* q^{n-k} *syndromes occur as syndromes of some vectors.*

Proof. Consider an (n,k) code C over $GF(q)$ with parity check matrix, $H=\begin{bmatrix}h_1\\h_2\\\vdots\\h_{n-k}\end{bmatrix}$. First, we shall prove that different vectors of the same coset have the same syndrome. Consider a coset $a+C$ of C. Let $a+c_1, a+c_2 \in a+C$. Then,

$$Syn(a+c_1)=\begin{bmatrix}(a+c_1).h_1\\(a+c_1).h_2\\\vdots\\(a+c_1).h_{n-k}\end{bmatrix}=\begin{bmatrix}ah_1+c_1h_1\\ah_2+c_1h_2\\\vdots\\ah_{n-k}+c_1h_{n-k}\end{bmatrix}=\begin{bmatrix}ah_1+0\\ah_2+0\\\vdots\\ah_{n-k}+0\end{bmatrix},$$

since every code word is orthogonal to every row of H. Similarly, $Syn(a+c_2)=\begin{bmatrix}ah_1\\ah_2\\\vdots\\ah_{n-k}\end{bmatrix}$.

Hence, vectors in same cosets have the same syndrome.

Next, we shall prove that vectors of different cosets have different syndromes. Let $a+C$ and $b+C$ be two distinct cosets of C. Let $a+c_1 \in a+C$ and $b+c_2 \in b+C$. Then,

$$\begin{aligned}
&\mathrm{Syn}(a+c_1) = \mathrm{Syn}(b+c_2) \\
&\Rightarrow \begin{bmatrix} ah_1 \\ ah_2 \\ \vdots \\ ah_{n-k} \end{bmatrix} = \begin{bmatrix} bh_1 \\ bh_2 \\ \vdots \\ bh_{n-k} \end{bmatrix} \\
&\Rightarrow a.h_i = b.h_i, \forall i \\
&\Rightarrow (a-b).h_i = 0, \forall i \\
&\Rightarrow a-b \in C
\end{aligned}$$

which would imply a and b belong to the same coset, a contradiction. The last statement is obvious.

Theorem 1.3.4 *If C is a binary code and x is any vector, then the syndrome of x is obtained by summing the columns of the parity check matrix H corresponding to the positions where x has non-zero entries.*

Theorem 1.3.5 *If C has minimum weight d. Then any vector of weight $t = \left\lfloor \frac{d-1}{2} \right\rfloor$ or less is a unique coset leader.*

Proof. Consider a coset of C with a coset leader x of weight 1. Then $d(x,0) = 1$, where $0 \in C$ is the zero vector. Suppose $y \in C + x$ is another vector of weight 1. Then y is decoded to $y - x$. But, according to the definition, y can also be a coset leader and hence can be decoded to 0. This contradicts the fact that C with minimum weight d can correct up to $t = \left\lfloor \frac{d-1}{2} \right\rfloor$ errors if $y - x \neq 0$. The same argument follows for any coset with a coset leader of weight $\leq \left\lfloor \frac{d-1}{2} \right\rfloor$.

Theorem 1.3.6 *If C is a code of even weight d and $t = \left\lfloor \frac{d-1}{2} \right\rfloor$. Then there are two vectors of weight $t+1$ in some coset of C.*

Proof. When d is even, $t = \left\lfloor \frac{d-1}{2} \right\rfloor = \frac{d-2}{2}$, and thus $t+1 = \frac{d}{2}$. Since $t+1 = \frac{d}{2}$ is a code of weight d, there exists $u \in C$ of weight d. Let x be a vector of weight $\frac{d}{2}$, which agrees with u at $\frac{d}{2}$ non-zero components. Then the vector $y = x - u$ is of weight $\frac{d}{2}$. Further x an y belong to the same coset since $x - y \in C$.

Definition 1.3.7 An (n,k,d) *code is said to be perfect, if all the vectors in V are contained in spheres of radius $t = \left\lfloor \frac{d-1}{2} \right\rfloor$ about the code words.*

For example, the (7,4,3) Hamming code is a perfect code. It has 16 code words and can correct up to $t = \left\lfloor \frac{3-1}{2} \right\rfloor = 1$ error. And, since the length of each code word is 7, there are a total of $2^7 = 128$ vectors. For the same reason, every sphere of radius 1 about a code word will contain (1 + 7) vectors. Therefore, the 16 spheres will contain a total of 128 vectors, making it a perfect code. In contrast, the $(6,3,3)$ binary code C_3 is not perfect. The code consists of eight code words and a total of 2^6 vectors. Here, $t = 1$ and $n = 6$ make each sphere of radius 1 contain seven vectors. Thus, all spheres together will contain 56 vectors only.

Theorem 1.3.8 *In order for a perfect t error-correcting binary (n,k) code to exist, the numbers n,k, and t must satisfy,*

$$\left(nC_0 + nC_1 + \ldots + nC_t\right)2^k = 2^n$$

And, in order for a perfect t error-correcting (n,k) code over $GF(q)$ to exist, the numbers n,k, and t must satisfy,

$$\left(nC_0 + (q-1)nC_1 + (q-1)^2 nC_2 + \ldots + (q-1)^t nC_t\right)q^k = q^n$$

Proof. Consider a t error-correcting binary (n,k) code. Let x be a code word and $S_t(x)$ be the sphere of radius t about the code word x. Then $S_t(x)$ will consist of x and all vectors that differ from x up to t places. The number of vectors which differ from x at one place is nC_1, the number of vectors which differ from x at two places is nC_2, and so on. Therefore, the total number of vectors in $S_t(x)$ is $1 + nC_1 + nC_2 + \ldots + nC_t$. Thus, the 2^k spheres will contain a total of $(1 + nC_1 + nC_2 + \ldots + nC_t)2^k$ vectors. Therefore, the code is perfect iff the above value is 2^n.

The general case is similar to the previous case. Here, the number of vectors that differ from x at t places is $(q-1)^t \cdot nC_t$, with each place differing by $(q-1)$ different values.

Problem 1.3.9 *Show that there a single-error-correcting binary perfect code of length 6.*

We know that an (n,k) binary perfect single-error-correcting code exists if $(1+n)2^k = 2^n$. That is, $1 + n = 2^{n-k}$, a power of 2. If $n = 6$, LHS is 7, not a power of 2.

1.4 PACKING RADIUS, COVERING RADIUS

Definition 1.4.1 *The packing radius t is the largest of all values of s, where spheres of radius s about code words are disjoint.*

Theorem 1.4.2 *The packing radius t has the following properties.*

1. If C is of weight d, then $t = \left[\frac{d-1}{2}\right]$.
2. t is the largest among the numbers s so that each vector of weight $\leq s$ is a unique coset leader.

Proof.

1. Let $t' = \frac{d-1}{2}$. Then, spheres of radius up to t' about the code words are disjoint. It is enough to show that spheres of radius $t'+1$ need not be disjoint. There are two cases.

   Case 1. Let d be even. Then $t' = \left[\frac{d-1}{2}\right] = \frac{d-2}{2}$ and $t'+1 = \frac{d}{2}$. Let $u \in C$ be of weight d and x be a vector of weight $\frac{d}{2}$ that agrees with u at $\frac{d}{2}$ non-zero components of u. Now $d(x,0) = \frac{d}{2} = d(x,u)$. But $t'+1 = \frac{d}{2}$. Thus, $x \in S_{t'+1}(0) \cap S_{t'+1}(u)$.

   Case 2. Let d be odd. Then $t' = \left[\frac{d-1}{2}\right] = \frac{d-1}{2}$ and $t'+1 = \frac{d+1}{2}$. Let $u \in C$ be of weight d and x be a vector of weight $\frac{d+1}{2}$ that agrees with u at $\frac{d+1}{2}$ non-zero components of u. Now $d(x,0) = \frac{d+1}{2} = t'+1$ and $d(x,u) = d - \frac{d+1}{2} = t'$. Thus $x \in S_{t'}(u) \Rightarrow x \in S_{t'+1}(u)$, and hence $x \in S_{t'+1}(0) \cap S_{t'+1}(u)$.

   $\therefore$ From both cases $t = t'$.

2. It is known from Theorem 1.3.5 that any vector of weight $\leq t$ is a unique coset leader. It is enough to show that a vector of weight $t+1$ need not be a unique coset leader. The case when d is even clear from Theorem 1.3.6. Now, let d be odd and x be a vector of weight $t+1=\frac{d+1}{2}$ that agrees with u at $\frac{d+1}{2}$ non-zero components of u. Let $y=x-u$. Then $x-y\in C$, and thus x and y belong to the same coset. But $weight(y)=weight(x-u)=d-\frac{d+1}{2}=\frac{d-1}{2}<\frac{d+1}{2}$, which follows that x cannot be a coset leader.

Definition 1.4.3 *Covering radius of a code C is the least value of such that spheres of radius s about the code words contain all elements of V.*

Theorem 1.4.4 *The covering radius r of a code C has the following properties*:

1. *r is the weight of the coset of largest weight.*
2. *r is the smallest among the numbers so that every syndrome is a combination of s or fewer columns of any parity check matrix.*

Proof.

1. *Let* a be the weight of the coset leader of the coset of largest weight. We shall prove that $a=r$. Suppose $a>r$ and x be a coset leader such that $wt(x)=a$. Note that $x-c$ belongs to the same coset for every $c\in C$. since x is the coset leader,

$$wt(x-c)\geq wt(x)=a,\forall c\in C$$
$$\text{i.e.,}\ d(x,c)\geq a>r,\forall c\in C$$
$$\text{i.e.,}\ x\notin S_r(c),\forall c\in C$$

This is a contradiction to the fact that r is the covering radius of C.

$$\therefore a\leq r \tag{1.2}$$

Now, we claim that spheres of radius a centered about the code words cover the whole space. If not, $\exists$ some vector $y\in V$, which does not belong to any of the spheres of radius a centered at any code word.

$$\text{i.e.,}\ d(y,c)>a,\forall c\in C \tag{1.3}$$

Now y belongs to some coset, and suppose its coset leader is z. $\therefore y=z+c$ for some $c\in C$.

$$\therefore d(y,c)=d(z+c,c)=wt(z+c-c)=wt(z)\leq a \tag{1.4}$$

We arrive at a contradiction from (1.3) and (1.4). Since r is the smallest non-negative integer such that spheres of radius r about code words cover V, we get $r\leq a$. Combining with (1.2), we obtain $r=a$.

2. First, we claim that the syndrome of any vector of weight d is a linear combination of d columns of the parity check matrix of C. For, let $H=[P_1\ P_2\ \ldots\ P_n]$, where $P_1,P_2,\ldots,P_n$ are the columns of the parity check matrix H. If $y=(y_1,y_2,\ldots,y_n)$ is any vector of weight d, then exactly d components of y are non-zero numbers, and hence $Syn(y)=Hy^T=y_1P_1+y_2P_2+\cdots+y_nP_n$, where $y_i\in GF(q)$. Thus $Syn(y)$ is a linear combination of d columns of H.

$\therefore$ The syndrome of any coset leader of weight i is a linear combination of i columns of H. Since the syndrome of any vector x is the same as the syndrome of the coset leader

of the coset containing x, we get the syndrome of x is a linear combination of as many columns of H as the weight of the coset leader. Since r is the weight of the coset leader of the coset of largest weight, we get the syndrome of any vector is a linear combination of r or fewer columns of H, and r is the smallest number having this property.

Remark 1.4.5 *When $t = r$, that is, if the packing radius of C coincides with the covering radius, then the code is perfect. If $r > t$, the code is not perfect.*

Theorem 1.4.6 *If u is of weight s in a linear code C, then the s columns of any parity check matrix H corresponding to the non-zero positions of u are linearly dependent. Conversely, any linear dependence among s columns of H corresponds to a code word of weight s in C.*

Proof. Since u is of weight s and is orthogonal to each row of any parity check matrix H of C, the s non-zero components become the coefficients of the linear combination of the corresponding columns of H to yield the zero vector. Thus, there is a dependence relation with the s columns of H.

Conversely, any dependence relation of s columns of H corresponds to a vector in C whose components are the respective coefficients in the dependence relation. Clearly weight of the vector is s.

Remark 1.4.7 *This theorem gives some information about the minimum weight of a code C by inspecting the columns of its parity check matrix H.*

Corollary 1.4.8 *A code contains a vector of weight 1 iff its parity check matrix H contains a zero column.*

Proof. By the previous theorem, $\exists$ a code word of weight 1 iff $\exists$ a linear dependence relation of the corresponding column of H. And, a single column is linearly dependent iff it is a zero column.

Corollary 1.4.9 *Any binary code C is of weight ≥ 3.*

Proof. Clearly, the sum of two distinct columns of binary digits cannot be zero. $\therefore$ To obtain a linear dependence relation, at least three columns are necessary, which implies that the weight of C is ≥ 3.

Corollary 1.4.10 *A linear code C has weight d iff d is the largest number so that every $d-1$ column of any parity check matrix H of C is linearly independent.*

Proof. Since each code word is of weight $\geq d$, for any dependence relation among the columns of H at least d columns are necessary. That is, each set of $d-1$ or fewer columns is linearly independent. Since there exist code words of weight d, $\exists$ at least one dependence relation among d columns of H. Thus d is the largest number such that each set of $d-1$ columns of H is independent.

Conversely, the hypothesis implies that weight of C is $\geq d$. Again since d is the largest number satisfying the hypothesis, $\exists$ at least one set of d columns of H that are dependent, which yields a code word of weight d.

Corollary 1.4.11 *If a code C has weight d, then C can detect all errors of weight $\leq d-1$.*

Proof. Let y be any received vector such that its error is of weight $\leq d-1$. Then $y = c + e$ for some codeword c and $\text{wt}(e) \leq d-1$. $\therefore Syn(y) = Syn(c+e) = Syn(c) + Syn(e) = Syn(e) \neq 0$, since $Syn(e)$ is a linear combination of $d-1$ or fewer columns of a parity check matrix H of C. Since $Syn(y) \neq 0$, we detect that there is an error in y, under the assumption that every error is of weight $\leq d-1$.

Corollary 1.4.12 *If C is a code of minimum weight $2(t+1)$, then C can detect and correct all errors of weight $\leq t$ and detect all errors of weight $t+1$.*

Proof. By a previous theorem, we get C can detect and correct all errors of weight $\leq \left\lfloor \frac{2(t+1)-1}{2} \right\rfloor = t$. Consider the spheres of radius t centered at distinct code words. If r is any received vector with an error of weight $t+1$, then r cannot be contained in any of the spheres.

In any matrix A, the number of independent rows is equal to the number of independent columns. That is, for any matrix A, row rank of $A =$ column rank of A.

Corollary 1.4.13 *For any (n,k,d) linear code, n,k,d satisfy the inequality $n-k \geq d-1$.*

Proof. Let C be any (n,k,d) linear code and H be any parity check matrix of C. Then H is an $(n-k)\times n$ matrix.

$\therefore$ The number of linearly independent columns of H is $\leq n-k$.

Since the weight of C is d, by a previous theorem, any $d-1$ columns of H are linearly independent.

$\therefore$ The number of linearly independent columns of H is $\geq d-1$.

These two facts prove $n-k \geq d-1$.

1.5 MDS CODE

Definition 1.5.1 *An (n,k,d) linear code is called an MDS code or MDS code iff $k = n-d+1$.*

Theorem 1.5.2 *If C is an (n,k,d) linear code, then every $(n-d+1)$ coordinate position contains an information set. In addition, d is the largest number with this property.*

Proof. Suppose G is a generator matrix for C and S a set of $(n-d+1)$ coordinate positions. If possible, assume that S does not contain an information set, then we arrive at a contradiction as follows. Consider a matrix A whose columns are those columns of G labeled by the numbers in S.A is a $k\times(n-d+1)$ matrix whose rank is less than k by our assumption that no k columns are independent. Thus, there is a dependence relation among the rows of A, say, $r_1, r_2, \ldots, r_k$. There are scalars $c_1, c_2, \ldots, c_k$ so that $r_1c_1 + r_2c_2 + \ldots + r_kc_k = 0$. Let $g_1, g_2, \ldots, g_k$ be the rows of G and consider the vector $x = g_1c_1 + g_2c_2 + \ldots + g_kc_k$. Clearly x is in C and has zeros in all the positions in S so that $wt(x) \leq n-(n-d+1) = d-1$. This is a contradiction since the weight of C is d. Then every $n-d+1$ position contains an information set.

Now let u be a vector of weight d, and then the $n-d$ positions where u is zero cannot contain an information set, since the rank of the submatrix of G with these positions as columns is less than k.

Theorem 1.5.3 *Every k position of an MDS code is an information set.*

Proof. Let C be an (n,k,d) MDS code.

$$\therefore k = n-d+1 \tag{1.5}$$

By the previous theorem, we know that every $n-d+1$ position of an (n,k,d) code contains an information set. $\therefore$ From (1.5) we get that every k position of an (n,k,d) MDS code contains an information set.

Theorem 1.5.4 *Dual of an MDS code is an MDS code.*

Proof. Let C be an (n,k,d) MDS code.

$\therefore k = n - d + 1$

Let G be any generator matrix of C. Note that G is the parity check matrix of the dual code $C^{\perp}$. Since C is an MDS code, every k position of C is an information set. That is, k columns of G are linearly independent. Since G is a $k \times n$ matrix, any $k+1$ columns of G are linearly dependent. $\therefore$ By a previous theorem, the weight of $C^{\perp}$ is $k+1$. Thus, $C^{\perp}$ is an $(n, n-k, k+1)$ linear code. Since $n-k = n-(k+1)+1$, we get that $C^{\perp}$ is an MDS code.

Definition 1.5.5 *A code C is said to be self-orthogonal if $C \subseteq C^{\perp}$.*

We can easily verify that C_2 is self-orthogonal. The Hamming code is not self-orthogonal. But we can extend the generator matrix as follows to make the code self-orthogonal.

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

This code is referred to as the extended Hamming code. It is an (8,4) code.

Theorem 1.5.6 *If the rows of a generator matrix G for a binary (n,k) code C have even weight and are orthogonal to each other. Then C is self-orthogonal and conversely.*

Proof. Let $G = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix}$; u_i is the ith row of G. Since each u_i is of even weight, $u_i.u_i = 0$. Further given that $u_i.u_j = 0 \forall i$ and j. Consider two code words c_1 and c_2, and then $c_1 = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_k u_k$ and $c_2 = \beta_1 u_1 + \beta_2 u_2 + \ldots + \beta_k u_k$. Then, $c_1.c_2 = 0$, since $u_i.u_i = 0$ and $u_i.u_j = 0$. Therefore, each code word is orthogonal to itself and is orthogonal to the other code words. Thus C is self-orthogonal.

Conversely, if C is self-orthogonal, every row of its generator matrix is a code word. Hence, each row will be orthogonal to itself and to all other rows. Hence the proof.

Theorem 1.5.7 *If the rows of a generator matrix G for a ternary (n,k) code C have a weight which is divisible by 3 and is orthogonal to each other. Then C is self-orthogonal and conversely.*

Proof. Let $G = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix}$; u_i is the ith row of G. Suppose u_i has r_i number of 1's and s_i number of 2's. Then $r_i + s_i = 0$, since the weight of each u_i is divisible by 3. Thus, $u_i.u_i = 1 \times r_i + 4 \times s_i = r_i + s_i = 0$. In addition, given that $u_i.u_j = 0 \forall i$ and j. Consider two code words c_1 and c_2, then $c_1 = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_k u_k$ and $c_2 = \beta_1 u_1 + \beta_2 u_2 + \ldots + \beta_k u_k$. Also $c_1.c_2 = 0$, since $u_i.u_i = 0$ and $u_i.u_j = 0$. Therefore, each code word is orthogonal to itself and is orthogonal to other code words. Thus C is self-orthogonal.

Conversely, if C is self-orthogonal, then every row u_i of G is a code word. Hence, each row will be orthogonal to itself and to all other rows. In particular $u_i.u_i = 0$, which simply says that the weight of each u_i is divisible by 3.

Theorem 1.5.8 *If the rows of a generator matrix G for a binary* (n,k) *code C have a weight divisible by* 4 *and is orthogonal to each other. Then* C *is self-orthogonal and all code words have weight divisible by 4.*

4. Let $G = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_k \end{bmatrix}$. A code word is obtained by taking the linear combination of the rows of G. To prove our result, it is enough to show that $wt(g_i + g_j)$ is divisible by 4. We have $wt(g_i + g_j) = wt(g_i) + wt(g_j) - 2(g_i * g_j)$. Both $wt(g_i)$ and $wt(g_j)$ are divisible by 4. Further, since g_i is orthogonal to $g_j, g_i * g_j$ is even. Hence proved.

Proof. The first part of the theorem is evident from the previous theorem. The only thing we have to prove is that the weight of each code word is divisible by

1.6 CONSTRUCTION OF FINITE FIELDS HAVING 16 ELEMENTS

We construct a field with 16 elements using the 16 binary 4-tuples, where each 4-tuple is associated with a polynomial of degree at most 3 over the binary field. Thus, polynomials play a central role in our construction. Let us briefly recall some basic definitions. A polynomial $f(x)$ is an expression of the form $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, where the coefficients a_i belong to a field F (referred to as scalars). A polynomial is said to be monic if its leading coefficient $a_n = 1$, and it is irreducible if it cannot be factored into polynomials of lower degree over F, except for scalar multiples of itself and the unit element. In many ways, monic irreducible polynomials over a field play a role analogous to that of prime numbers in the integers.

Euclidean Algorithm for polynomials. If $a(x)$ and $b(x)$ are any two polynomials, then there exist polynomials $c(x)$ and $d(x)$ such that

$$g.c.d(a(x), b(x)) = a(x)c(x) + b(x)d(x)$$

Division Algorithm for polynomials. If $a(x)$ and $b(x)$ are two polynomials with $\deg(b(x)) < \deg(a(x))$, then there exist polynomials $q(x)$ and $r(x)$ such that

$$a(x) = b(x)q(x) + r(x); \deg(r(x)) < \deg(b(x))$$

Method to find GCD using division algorithm. Let $a(x)$ and $b(x)$ be two polynomials with $\deg(b(x)) \leq \deg(a(x))$; then by division algorithm there exist polynomials $q(x)$ and $r(x)$ such that

$$a(x) = b(x)q(x) + r(x); \deg(r(x)) < \deg(b(x)) \tag{1.6}$$

Again by division algorithm there exist $q_1(x)$ and $r_1(x)$ such that

$$b(x) = r(x)q_1(x) + r_1(x); \deg(r_1(x)) < \deg(r(x)) \tag{1.7}$$

Again by division algorithm there exist $q_2(x)$ and $r_2(x)$ such that

$$r(x) = r_1(x)q_2(x) + r_2(x); \deg(r_2(x)) < \deg(r_1(x)) \tag{1.8}$$

This cannot be continued infinitely. So let us assume that there exists $s_1(x)$ such that

$$
\begin{aligned}
& r_1(x) = r_2(x)s_1(x) \\
& \Rightarrow r_2(x) \mid r_1(x) \\
& \Rightarrow r_2(x) \mid r(x) \text{ by } (1.8) \\
& \Rightarrow r_2(x) \mid b(x) \text{ by } (1.7) \\
& \Rightarrow r_2(x) \mid a(x) \text{ by } (1.6) \\
& \therefore r_2(x) \mid a(x) \text{ and } r_2(x) \mid b(x)
\end{aligned}
$$

Now suppose $s(x)$ divides both $a(x)$ and $b(x)$. Then by (1.6) $s(x)$ divides $r(x)$, and hence by (1.7) $s(x)$ divides $r_1(x)$, and by (1.8), it also divides $r_2(x)$. Therefore $r_2(x)$ is the g.c.d. of $a(x)$ and $b(x)$.

Theorem 1.6.1 *If $f(x)$ is a polynomial over a field F. Then α is a root of $f(x)$ iff $f(x) = (x-\alpha)g(x)$.*

Proof. Let α be a root of $f(x)$. Then there exist polynomials $g(x)$ and $r(x)$ such that

$$f(x) = g(x)(x-\alpha) + r(x); \deg\big(r(x)\big) < \deg(x-\alpha)$$

This implies $r(x)$ is a constant. But $f(\alpha) = 0$ implies $r(\alpha) = 0$. Thus, $r(x)$ is nothing but the zero polynomial. $\therefore f(x) = (x-\alpha)g(x)$. Converse is trivial.

Theorem 1.6.2 *If an irreducible monic polynomial $f(x)$ divides $g(x)h(x)$, then either $f(x) / g(x)$ or $f(x) / h(x)$.*

Proof. Let $f(x)$ divide $g(x)h(x)$. If possible, assume that $f(x) \nmid g(x)$. Then, g.c.d. $\big(f(x), g(x)\big) = 1$ and by the Euclidean algorithm, $\exists c(x), d(x)$ such that

$$1 = g.c.d\big(f(x), g(x)\big) = f(x)c(x) + g(x)d(x)$$

Multiplying both sides by $h(x)$,

$$h(x) = f(x)c(x)h(x) + g(x)d(x)h(x)$$

Since $f(x)$ divides R.H.S., $f(x)$ divides $h(x)$. Hence the proof.

Theorem 1.6.3 *Every monic polynomial over a field F can be expressed uniquely as a product of irreducible polynomials over F.*

Proof. If $f(x)$ is irreducible, then there is nothing to prove. Let us assume that $f(x)$ is reducible. Then, $\exists$ polynomials $g(x), h(x)$ such that

$$f(x) = g(x)h(x) \text{ and } \deg\big(g(x)\big), \deg\big(h(x)\big) < \deg f(x)$$

If $g(x)$ and $h(x)$ are both irreducible, then we are done. If either or both of them are reducible, then proceed as follows. Suppose $g(x)$ is reducible. Then, $\exists$ polynomials $g_1(x), h_1(x)$ such that

$$g(x) = g_1(x)h_1(x) \text{ and } \deg\big(g_1(x)\big), \deg\big(h_1(x)\big) < \deg\big(g(x)\big)$$

Then, $f(x) = g_1(x)h_1(x)h(x)$. We cannot continue this infinitely since $\deg(f(x))$ is finite. So, after a finite number of repetitions, we reach a stage where $f(x)$ is expressed as a product of irreducible polynomials. Since $f(x)$ is monic, the irreducible factors are also monic.

To prove the uniqueness, without loss of generality, let us assume that $f(x)$ can be expressed as a product of two irreducible factors in two different ways, say, $f(x) = g_1(x)h_1(x)$ and $f(x) = g_2(x)h_2(x)$. Then, $g_2(x)$ divides $g_1(x)h_1(x)$. Since $g_2(x)$ is irreducible, it either divides $g_1(x)$ or $h_1(x)$. But $g_1(x), h_1(x)$ are irreducible, which implies $g_2(x) = g_1(x)$ or $g_2(x) = h_1(x)$. For the former, we get $h_2(x) = h_1(x)$ and for the latter, $h_2(x) = g_1(x)$. In both cases, the expression becomes unique.

Theorem 1.6.4 *If $f(x)$ is a polynomial over a field of degree n. Then $f(x)$ has at most n distinct roots.*

Proof. Without loss of generality, assume that $f(x)$ is monic. The result can be proved by induction on n. For $n = 1, f(x)$ is having exactly one root.

For $n = 2$, if $f(x)$ is irreducible, then there are no roots. Otherwise, $\exists\, \alpha$ such that α is a root of $f(x)$

$$\begin{aligned}
&\therefore f(x) = (x-\alpha)g(x)\\
&\Rightarrow g(x) \text{ is monic and linear}\\
&\Rightarrow g(x) = (x-\alpha) \text{ or } (x-\beta)\\
&\Rightarrow f(x) = (x-\alpha)^2 \text{ or } (x-\alpha)(x-\beta)
\end{aligned}$$

Suppose the result holds for all polynomials of degree $\leq n-1$. Now, let $f(x)$ be of degree n. Then $f(x)$ is irreducible or $\exists \alpha$ such that α is a root of $f(x)$. Then, $f(x) = (x-\alpha)g(x)$, with $g(x)$ having at most $n-1$ distinct roots. Thus, $f(x)$ has at most n distinct roots.

Definition 1.6.5 *A polynomial $a(x)$ is said to be congruent to $b(x) \bmod f(x)$ if $f(x)$ divides $a(x) - b(x)$.*

We write this, as $a(x) \equiv b(x) \bmod f(x)$. By the division algorithm, for the given polynomials $a(x), f(x)$, there exists $c(x), d(x)$ such that $a(x) \equiv c(x)f(x) + b(x); \deg(b(x)) < \deg(f(x))$. Then $a(x) \equiv b(x) \bmod f(x)$ and we can easily verify that $b(x)$ is a polynomial of the least degree, so that the above is true.

Corollary 1.6.6 *If $a(x) \equiv c(x) \bmod f(x)$ and $b(x) \equiv d(x) \bmod f(x)$. Then,*

1. $a(x) \pm b(x) \equiv c(x) \pm d(x) \bmod f(x)$
2. $a(x)b(x) \equiv c(x)d(x) \bmod f(x)$

Proof.

1. Given $a(x) \equiv c(x) \bmod f(x)$ and $b(x) \equiv d(x) \bmod f(x)$. So, $\exists g(x), h(x)$ such that $a(x) = g(x)f(x) + c(x)$ and $b(x) = h(x)f(x) + d(x)$.

$$\begin{aligned}
\therefore a(x) \pm b(x) &\equiv g(x)f(x) \pm h(x)f(x) + c(x) \pm d(x)\\
&\equiv \big(g(x) \pm h(x)\big)f(x) + \big(c(x) \pm d(x)\big)
\end{aligned}$$

2. Substituting for the values of $a(x)$ and $b(x)$,

$$\begin{aligned} a(x)b(x) &= \big[g(x)f(x)+c(x)\big]\big[h(x)f(x)+d(x)\big] \\ &= \big[g(x)h(x)f(x)+g(x)h(x)+c(x)h(x)\big]f(x)+\big[c(x)d(x)\big] \\ &\equiv c(x)d(x) \bmod f(x) \end{aligned}$$

Corollary 1.6.7 *If $f(x)$ is an irreducible polynomial of deg n and $a(x)$ is a polynomial of deg $\leq n-1$, then there exists a polynomial $b(x)$ of deg $\leq n-1$ such that $a(x)b(x) \equiv 1 \bmod f(x)$.*

Proof. Since $f(x)$ is irreducible, g.c.d. $\big(a(x), f(x)\big) = 1$, and by the Euclidean algorithm, $\exists b(x), c(x)$ such that $a(x)b(x)+c(x)f(x) = 1$. Then $a(x)b(x) \equiv 1 \bmod f(x)$.

If $\deg\big(b(x)\big) > \deg\big(f(x)\big)$, then we can find a unique polynomial $b_1(x)$ such that $\deg\big(b_1(x)\big) < \deg\big(f(x)\big)$ and $b(x) \equiv b_1(x) \bmod f(x)$.

$$\begin{aligned} &\text{i.e. } b(x) - b_1(x) = d(x)f(x), \text{ for some } d(x) \\ &\therefore a(x)b(x) = a(x)b_1(x) + a(x)d(x)f(x) \\ &\qquad\qquad \equiv a(x)b_1(x) \bmod f(x) \end{aligned}$$

Combining this with the previous equation, we get $a(x)b_1(x) \equiv 1 \bmod f(x)$, where $b_1(x)$ is a unique polynomial such that $\deg\big(b_1(x)\big) < \deg\big(f(x)\big)$.

Now we are ready with our goal for the construction of a field of 16 elements. Let us start with a set $\mathbb{F}$ of all binary four-tuples. We shall associate with each of the 16 binary four-tuples a polynomial, as follows:

Four-tuple				*Polynomial*
0	0	0	0	0
0	0	0	1	1
0	0	1	0	x
0	0	1	1	$x+1$
0	1	0	0	x^2
0	1	0	1	x^2+1
0	1	1	0	x^2+x
0	1	1	1	x^2+x+1
1	0	0	0	x^3
1	1	0	0	x^3+x^2
1	1	1	0	x^3+x^2+x
1	0	1	0	x^3+x
1	0	1	1	x^3+x+1

1	1	0	1	$x^3 + x^2 + 1$
1	0	0	1	$x^3 + 1$
1	1	1	1	$x^3 + x^2 + x + 1$

We can see that all the binary polynomials of degree < 4 are associated with different elements of $\mathbb{F}$ in a unique manner. Further $\mathbb{F}$ is clearly a group under addition modulo 2. But it is not closed under multiplication. To accomplish this, we take multiplication modulo $f(x) = x^4 + x + 1$, an irreducible polynomial. Now, $\mathbb{F}$ has become a set closed under multiplication, associative under multiplication, has a multiplicative identity Viz $1(0\ \ 0\ \ 0\ \ 1)$ and also satisfies the distributive property. It is enough to show that every non-zero element of $\mathbb{F}$ has a multiplicative inverse $\bmod f(x)$. Consider any non-zero element of $\mathbb{F}$. Let $a(x)$ be its associated polynomial. Then $\deg(a(x)) < \deg(f(x)) = 4. \therefore$ g.c.d $(a(x), f(x)) = 1$.

Then by the Euclidean algorithm, $\exists\, b(x), c(x)$ such that $a(x)b(x) + c(x)f(x) = 1$.

That is, $a(x)b(x) \equiv 1 \bmod f(x)$

If $\deg(b(x)) > \deg(f(x))$. Then we can find $b_1(x)$ such that

$$b(x) \equiv b_1(x) \bmod f(x) \text{ and } \deg b_1(x) \leq \deg f(x)$$
$$\therefore a(x)b(x) \equiv a(x)b_1(x) \equiv 1 \bmod f(x)$$

That is, $a(x)$ has a multiplicative inverse modulo $f(x)$ in the above list of polynomials. The binary four-tuple associated with this will be the corresponding multiplicative inverse. To facilitate this computation, we shall express every polynomial as a power of α, with α being a root of $f(x) = x^4 + x + 1$. Then, $\alpha^4 + \alpha + 1 = 0$, and hence $\alpha^4 = \alpha + 1 \Rightarrow \alpha^5 = \alpha^2 + \alpha \Rightarrow \alpha^6 = \alpha^3 + \alpha^2$, and so on. The following table gives the expression of each four-tuple in terms of power of α.

Four-tuple				*Power of* α
0	0	0	0	0
0	0	0	1	1
0	0	1	0	α
0	1	0	0	α^2
1	0	0	0	α^3
0	0	1	1	α^4
0	1	1	0	α^5
1	1	0	0	α^6
1	0	1	1	α^7
0	1	0	1	α^8
1	0	1	0	α^9
0	1	1	1	α^{10}
1	1	1	0	α^{11}

1	1	1	1	α^{12}
1	1	0	1	α^{13}
1	0	0	1	α^{14}

Proceeding like this, we can see that $\alpha^{15} = 1$; that is, α^{15} corresponds to the multiplicative identity. This fact facilitates the computation of the multiplicative inverse for any non-zero element of $\mathbb{F}$. We shall demonstrate this by taking an example. Suppose we want to compute the inverse of $(0\ \ 1\ \ 1\ \ 0)$; the corresponding power of α is α^5. The inverse of α^5 is α^{10}, whose four-tuple is $(0\ \ 1\ \ 1\ \ 1)$. Thus, the inverse of $(0\ \ 1\ \ 1\ \ 0)$ is $(0\ \ 1\ \ 1\ \ 1)$. Thus, we have constructed a field of 16 elements, which is denoted by $GF(16)$.

Problem 1.6.8 *Perform the following operations in* $GF(16)$.

1. $1(1\ \ 0\ \ 0\ \ 1)\cdot(1\ \ 0\ \ 1\ \ 1)+(0\ \ 1\ \ 0\ \ 1)\div(1\ \ 1\ \ 0\ \ 0)$
2. $(1\ \ 1\ \ 1\ \ 0)^{\frac{1}{2}}+(1\ \ 1\ \ 0\ \ 1)$

Proof.

1. $(1\ \ 0\ \ 0\ \ 1).(1\ \ 0\ \ 1\ \ 1)=\alpha^{14}.\alpha^7=\alpha^{21}=\alpha^6=(1\ \ 1\ \ 0\ \ 0)$ and

$(0\ \ 1\ \ 0\ \ 1)\div(1\ \ 1\ \ 0\ \ 0)=\alpha^8\div\alpha^6=\alpha^2=(0\ \ 1\ \ 0\ \ 0)$

$\therefore(1\ \ 0\ \ 0\ \ 1).(1\ \ 0\ \ 1\ \ 1)+(0\ \ 1\ \ 0\ \ 1)\div(1\ \ 1\ \ 0\ \ 0)$
$=(1\ \ 1\ \ 0\ \ 0)+(0\ \ 1\ \ 0\ \ 0)$
$=(1\ \ 0\ \ 0\ \ 0)$

2. $(1\ \ 1\ \ 1\ \ 0)^{\frac{1}{2}}=\left(\alpha^{11}\right)^{\frac{1}{2}}=\left(\alpha^{26}\right)^{\frac{1}{2}}=\alpha^{13}=(1\ \ 1\ \ 0\ \ 1)$

$\therefore(1\ \ 1\ \ 1\ \ 0)^{\frac{1}{2}}+(1\ \ 1\ \ 0\ \ 1)=(1\ \ 1\ \ 0\ \ 1)+(1\ \ 1\ \ 0\ \ 1)=(0\ \ 0\ \ 0\ \ 0)$

Problem 1.6.9 *Construct another field of 16 elements using the irreducible polynomial* $f(x)=x^4+x^3+1$.

The construction is exactly the same as above. Here α is a root of x^4+x^3+1.

Four-tuple				*Power of* α
0	0	0	0	0
0	0	0	1	1
0	0	1	0	α
0	1	0	0	α^2
1	0	0	0	α^3
1	0	0	1	α^4
1	0	1	1	α^5
1	1	1	1	α^6

0	1	1	1	α^7
1	1	1	0	α^8
0	1	0	1	α^9
1	0	1	0	α^{10}
1	1	0	1	α^{11}
0	0	1	1	α^{12}
0	1	1	0	α^{13}
1	1	0	0	α^{14}

Result 1.6.10 *If* $a(x)$ *is a binary polynomial, then* $a(x)^2 = a\left(x^2\right)$.

Proof. Let $a(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n; a_i \in \{0,1\}$. Then,

$$\begin{aligned}(a(x))^2 &= \left(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n\right)^2 \\ &= a_0^2 + a_1^2x^2 + a_2^2\left(x^2\right)^2 + \ldots + a_n^2\left(x^n\right)^2 + 2\left(a_0a_1x + a_0a_2x^2 + \ldots + a_{n-1}x^{n-1}a_nx^n\right) \\ &= a_0 + a_1x^2 + a_2\left(x^2\right)^2 + \ldots + a_n\left(x^n\right)^2 + 0, \text{ since } a_i^2 = a_i, \forall i \\ &= a\left(x^2\right)\end{aligned}$$

1.7 BCH CODES

Here, we try to construct a double-error-correcting code of length 15. Let $H = \begin{bmatrix} 1 & 2 & 3 & \cdots & 15 \\ f(1) & f(2) & f(3) & \cdots & f(15) \end{bmatrix}$, where each positive integer $i \le 15$ represents the four-tuple corresponding to i and $f(i)$ represents the cube of i that is computed using the previous table for a field of 16 elements. We know that every element of a field of 16 elements constructed by taking $\bmod f(x) = x^4 + x^3 + 1$ can be expressed as a power of α, where α is a root of $f(x)$. Using this, H may be rewritten as $H = \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{14} \\ 1 & \alpha^3 & \alpha^6 & \cdots & \alpha^{12} \end{bmatrix}$ with a change in the order of the columns. Now suppose that no error occurs when a code word x is sent and received as y. Then $Syn(y) = 0$. Suppose a single error occurs in the $(i+1)^{th}$ position. Then $Syn(y) = \begin{bmatrix} \alpha^i \\ \alpha^{3i} \end{bmatrix}$. Further, suppose that double errors occur at the $(i+1)^{th}$ and $(j+1)^{th}$ position, then $Syn(y) = \begin{bmatrix} \alpha^i + \alpha^j \\ \alpha^{3i} + \alpha^{3j} \end{bmatrix}$. Now, let $\alpha^i + \alpha^j = y_1$ and $\alpha^{3i} + \alpha^{3j} = y_3$. By substituting these in the identity $a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$, we get $y_3 = y_1\left(y_2 - \alpha^i\alpha^j\right)$, where $y_2 = y_1^2$.

$\therefore \frac{y_3}{y_1} - y_2 = -\alpha^i\alpha^j$. Since we are dealing with binary four-tuples, we have $\frac{y_3}{y_1} + y_2 = \alpha^i\alpha^j$. It follows that α^i, α^j are the roots of $x^2 + y_1 x + \left(\frac{y_3}{y_1} + y_2\right) = 0$. Solving this equation, we can find α^i and α^j. To conclude, we can explain the BCH algorithm as follows:

Suppose y is received. Compute $Syn(y) = \begin{bmatrix} y_1 \\ y_3 \end{bmatrix}$.

1. If $y_1 = 0, y_3 = 0$, then there is no error.
2. If $y_1 = \alpha^i, y_3 = \alpha^{3i}$. Then a single error occurs at the $(i+1)^{th}$ position.
3. If the above are not happening, then solve the equation $x^2 + y_1 x + \left(\frac{y_3}{y_1} + y_2\right) = 0$. If α^i and α^j are the roots of this, then two errors occur at the $(i+1)^{th}$ and $(j+1)^{th}$ positions; otherwise, we can conclude that more than two errors occur.

Problem 1.7.1 *Express the parity check matrix for BCH codes as a binary matrix.*

We have $H = \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{14} \\ 1 & \alpha^3 & \alpha^6 & \cdots & \alpha^{12} \end{bmatrix}$

$$= \begin{bmatrix} 0&0&0&1&1&1&1&0&1&0&1&1&0&0 \\ 0&0&1&0&0&0&1&1&1&1&0&1&0&1 \\ 0&1&0&0&0&1&1&1&1&0&1&0&1&1 \\ 1&0&0&0&1&1&1&1&0&1&0&1&1&0 \\ 0&1&1&0&0&0&1&1&0&0&0&1&1&0 \\ 0&0&1&1&0&0&0&1&1&0&0&0&1&1 \\ 0&0&1&0&1&0&0&1&0&1&0&0&1&0 \\ 1&0&1&1&1&1&0&1&1&1&1&0&1&1 \end{bmatrix}$$

Problem 1.7.2 *Using the double-error-correcting BCH code, find the position of errors for x and y, where $Syn(x) = \begin{bmatrix} \alpha^{11} \\ \alpha^{14} \end{bmatrix}$ and $Syn(y) = \begin{bmatrix} \alpha^6 \\ \alpha^3 \end{bmatrix}$.*

First consider x. Since $\text{Syn}(x) = \begin{bmatrix} \alpha^{11} \\ \alpha^{14} \end{bmatrix}$ and $\alpha^{14} \neq \left(\alpha^{11}\right)^3$, more than one error has occurred. Here $y_1 = \alpha^{11}\left(\therefore y_1^2 = \alpha^7\right)$ and $y_3 = \alpha^{14}$. Substituting in the equation

$$x^2 + y_1 x + \left(\frac{y_3}{y_1} + y_2\right) = 0$$
$$\Rightarrow x^2 + \alpha^{11} x + \left(\alpha^3 + \alpha^7\right) = 0$$
$$\Rightarrow x^2 + \alpha^{11} x + \alpha^6 = 0$$

Solving this we can see that α^2 and α^4 are the roots (this is done by finding the two factors of α^6 whose sum is α^{11}). Therefore, the errors have occurred at the third and fifth positions.

Now consider y. Since $\text{Syn}(y) = \begin{bmatrix} \alpha^6 \\ \alpha^3 \end{bmatrix}$, here $\alpha^3 = \left(\alpha^6\right)^3$. Therefore a single error has occurred at the $(6+1)^{th} = 7^{th}$ position.

Problem 1.7.3 *Using the double-error-correcting BCH code, decode the following received vector* $x = (1\ \ 1\ \ 0\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 1\ \ 0\ \ 0\ \ 1)$.

We have $\text{syn}(x) = \begin{bmatrix} 1+\alpha+\alpha^3+\alpha^4+\alpha^5+\alpha^6+\alpha^8+\alpha^{10}+\alpha^{11}+\alpha^{14} \\ 1+\alpha^3+\alpha^9+\alpha^{12}+1+\alpha^3+\alpha^9+1+\alpha^3+\alpha^{12} \end{bmatrix} = \begin{bmatrix} \alpha^{12} \\ \alpha^4 \end{bmatrix}$. Here, $\alpha^4 \neq \left(\alpha^{12}\right)^3 = \alpha^6$. Therefore, considering the equation $x^2 + y_1 x + \left(\frac{y_3}{y_1} + y_2\right) = 0$ and substituting values for y_1, y_2, y_3, we get

$$x^2 + \alpha^{12}x + \left(\alpha^7 + \alpha^9\right) = 0$$
$$\Rightarrow x^2 + \alpha^{12}x + \alpha = 0$$

Therefore, the roots are α^0 and α^1, meaning the errors are in the first and second positions. Thus, the sent vector is $x' = [0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 1\ \ 0\ \ 0\ \ 1]$.

1.8 CONCLUSION

In this chapter, we explored the fundamental aspects of coding theory, focusing on error-detecting and error-correcting techniques. Starting with the Hamming (7,4) code, we examined the construction and decoding mechanisms of linear codes, with special attention to the roles of dual codes and the weight of code words. The decoding strategies, including maximum likelihood decoding and syndrome decoding, were discussed in detail, highlighting their practical relevance in minimizing transmission errors. We also introduced the concept of MDS codes, which achieve the theoretical limit of error correction for a given code length and dimension. The construction of a finite field with 16 elements served as a foundation for understanding more advanced codes like BCH codes, which offer powerful multierror correction capabilities. Through theoretical explanations and illustrative examples, this chapter provides a comprehensive introduction to classical coding techniques that are essential in digital communication and data storage.

1.9 FUTURE SCOPE

While this chapter covered foundational codes and decoding methods, the field of coding theory continues to evolve with emerging demands in high-speed communication and data security. Future research can explore modern coding schemes such as LDPC codes and Turbo codes, which offer near-capacity performance. In addition, the application of algebraic geometry codes and quantum error-correcting codes presents promising directions for secure and fault-tolerant communication systems. Further exploration of efficient decoding algorithms, field extensions, and their implementation in real-time systems could greatly enhance the performance of error control in practical applications.

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Chapter 2

Skew Generalized Polycyclic Codes with Derivations

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2.1 INTRODUCTION

Coding theory mainly studies the methods that are efficient and preserve the accuracy of information during transmission from one place to another. In coding theory, linear codes have been studied for the last seven decades. Initially, linear codes were studied over the binary field. In 1972, Blake (1) initiated the study of codes over finite rings. Cyclic codes are the most important class of linear codes from an implementation point of view and also play a prominent role in the development of algebraic coding theory. These codes have been studied over several finite rings and produce many new and optimal codes [1–5].

In 2007, Boucher et al. (6) introduced skew cyclic codes, which are a generalization of cyclic codes over a noncommutative ring, namely, the skew polynomial ring $\mathbb{F}_q[x;\theta]$, where $\mathbb{F}_q$ is a finite field and θ is an automorphism of $\mathbb{F}_q$. Besides the commutative case, skew polynomial rings have many applications in constructing algebraic codes with better parameters (7). It is well-known that the factorization of a polynomial plays a key role in studying these codes. More factorization leads to a rich algebraic structure that is capable of producing codes with good parameters. Therefore, it is logical to extend these codes over a structure in which polynomials have more factorization than the commutative structure. One smart choice is the skew polynomial ring. We always have more choices to construct new codes from the codes over the skew polynomial ring than in a commutative ring. That is one of the major motivations to study codes over a skew polynomial ring.

In 2009, Boucher and Ulmer (8) obtained some skew codes with a Hamming distance more significant than the known linear codes with the same code rate parameters. This work was further advanced by Boulagouaz and Leroy in 2013 with the development of (σ,δ) codes (9), and by Boucher and Ulmer (10) in 2014, Ma et al. (11) in 2021, Patel and Prakash (12) in 2022, who explored linear codes through the framework of skew polynomial rings with non-trivial automorphisms and inner derivations over finite fields. Later on, in 2017, Cuitiño and Tironi (13) studied the structural properties of skew generalized cyclic codes over finite fields and also obtained some Bose–Chaudhuri–Hocquenghem (BCH)-type lower bounds for their minimum distances, which generalizes the results of Hartmann and Tzeng (14) in the noncommutative case.

Moreover, cyclic, constacyclic, and negacyclic codes are part of a larger class of codes called polycyclic codes, that is, codes that can be viewed as ideals of a factor ring $\mathbb{F}_q[x]/(f)$, where f is some polynomial of degree n. They reduce to cyclic codes when $f = x^n - 1$ and to constacyclic codes when $f = x^n - a$, for some $a \in \mathbb{F}_q^*$. These codes can be seen as ideals of a factor ring $\frac{\mathbb{F}_q[z]}{\langle e(z)\rangle}$, where $e(z)$ is an n-degree polynomial. If $e(z) = z^n - b$, then it turns out to be a constacyclic for a unit $b \in \mathbb{F}_q^*$, and cyclic code when $b = 1$ [15–27]. The generalization of cyclic and constacyclic codes, polycyclic codes, were initially investigated

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by Peterson (18) (page 241) and were formerly known as pseudo-cyclic codes. In their study of the dual of a polycyclic code, Lopez-Permouth et al. (16) established that a constacyclic code is a polycyclic code, and its dual is again polycyclic. Since the dual of a polycyclic code is not necessarily polycyclic. Therefore, later these codes had been renewed by Alahmadi et al. (15) in terms of structural properties by adopting the techniques of the linear algebraic theory of invariant subspaces. Authors introduced an alternate duality (an alternate form of dual that is the annihilator of the ideal) to address this issue. In 2013, Lopez-Permouth et al. (19) introduced the concept of repeated-root constacyclic codes (when length n is not coprime to the characteristic of the ring) using polycyclic codes over Galois rings. These codes were also considered over finite chain rings in Martínez-Moro et al. (20). Thus, skew polycyclic codes are the most recent powerful generalization of all such codes in this sequel. Recently, Patel and Prakash (21) generalized results of Alahmadi et al. (15) from polycyclic codes to skew polycyclic codes in the setting of the ore polynomial ring $\mathbb{F}_q[z;\rho]$. In this chapter, we generalize these codes to two-dimensional skew polycyclic codes.

The two-dimensional cyclic code was introduced by Ikai et al. (22) in 1975 and Imai (23) in 1977. The two-dimensional theory is beneficial for the analysis and generation of two-dimensional periodic arrays. It gives a construction method for the two-dimensional feedback shift register with a minimum number of storage devices that generate a given two-dimensional periodic array. Simple 2-D linear feedback shift registers and combinatorial logic in digital VLSI circuits can be used to implement TDC codes. In (24; 25), 2-D codes for particular error patterns were examined. In 2014, Xiuli and Hongyan (26) generalized the concept as a two-dimensional skew cyclic code over a finite field. In 2019, Sharma and Bhaintwal (27) studied the structural behavior of a two-dimensional skew cyclic code over the ring $\mathbb{F}_q + u\mathbb{F}_q$. In (28), Sepasdar and Khashyarmanesh studied the algebraic structure of two-dimensional cyclic codes of length $n = s2^k$ over the finite field. This technique was later expanded to (λ_1, λ_2)-constacyclic codes by Prakash and Patel (29). Authors examined two-dimensional double cyclic codes over finite fields in (30). Other scholars had expressed interest in studying these codes (20, 31, 32). These works motivate us to study the structural properties of the two-dimensional skew polycyclic codes. Last but not least, BCH-type lower bounds for a minimum distance of skew generalized polycyclic codes with non-zero derivations are also considered, which generalizes the results of (13; 14).

The remainder of the chapter is organized as follows. Section 2.2 contains some preliminaries, while Section 2.3 discusses skew generalized polycyclic codes and their structural properties. Section 2.4 investigates BCH lower bounds for a minimum distance of skew generalized polycyclic codes with non-zero derivations. A sufficient condition for a code to be a maximum distance separable (MDS) code is also derived. Section 2.5 includes some examples of MDS codes and a comparative survey of skew polycyclic codes versus polycyclic codes. And, finally, section 2.6 concludes the chapter.

2.2 PRELIMINARIES

Recall that a linear code $\mathcal{L}$ of length n over a ring R is considered as an R-submodule of R^n and each member of $\mathcal{L}$ is a codeword. The rank of $\mathcal{L}$ is the minimum number of generators for $\mathcal{L}$ as a module, and the free rank of $\mathcal{L}$ is the rank of $\mathcal{L}$ if $\mathcal{L}$ is free. A linear code of length n and rank k is denoted by $[n,k]$. The Hamming weight for a codeword $c = (c_1, c_2, \ldots, c_n)$, is denoted by $w_H(c)$ and defined by $w_H(c) = |\{i : c_i \neq 0\}|$. Now, the Hamming distance between two codewords c' and c'' is defined as $d_H(c', c'') = w_H(c' - c'')$, and for $\mathcal{L}$, it is $d_H(\mathcal{L}) = \min\{d_H(c', c'') : c' \neq c'', c', c'' \in \mathcal{L}\}$. Therefore, an $[n,k]$ linear code

is compactly represented by $[n,k,d_H]$, where d_H is the Hamming distance. Also, the Euclidean inner product "." between any two vectors $c'=(c_1',c_2',\ldots,c_n')$ and $c''=(c_1'',c_2'',\ldots,c_n'')$ of R^n is defined by $c'\cdot c''=\sum_{i=1}^{n}c_i'c_i''$. It is evident that the (Euclidean) dual of $\mathcal{L}$ defined by $\mathcal{L}^{\perp}=\{c'\in R^n : c'\cdot c''=0, \text{for all } c''\in\mathcal{L}\}$ is a linear code when $\mathcal{L}$ is linear. In addition, $\mathcal{L}$ is said to be a *self-orthogonal* code if $\mathcal{L}\subseteq\mathcal{L}^{\perp}$, and a *self-dual* code if $\mathcal{L}=\mathcal{L}^{\perp}$.

If $\mathcal{L}$ is an $[n,k,d]$ code, then, from the Singleton bound, its minimum distance is bounded above by $d\le n-k+1$. A code that satisfies the above bound with equality is called *MDS*. A code is *almost MDS* if its minimum distance is one unit less than the MDS bound and *optimal* if it has the highest possible minimum distance for a given length and dimension.

Definition 2.2.1 *Let $\mathcal{R}$ be a finite ring with unity and τ be an automorphism of $\mathcal{R}$. Then a map $\delta_\tau:\mathcal{R}\to\mathcal{R}$ is said to be a τ-derivation of $\mathcal{R}$ if*

(a) $\delta_\tau(r+s)=\delta_\tau(r)+\delta_\tau(s)$;
(b) $\delta_\tau(rs)=\delta_\tau(r)s+\tau(r)\delta_\tau(s)$.

for all $r,s\in\mathcal{R}$.

Let $\mathbb{F}_q$ be a finite field of order $q=p^m$, where m is a positive integer. Suppose ρ is an automorphism of $\mathbb{F}_q$ defined by $\rho(a)=a^{p^t}$ and $\Im_\rho$ is a ρ-derivation of $\mathbb{F}_q$, where t is a positive integer. Further, the order of automorphism ρ is given by $|\langle\rho\rangle|=\frac{m}{\gcd(m,t)}$, and, in particular, $|\langle\rho\rangle|=\frac{m}{t}$ when $t\mid m$. We consider the set

$$\mathbb{F}_q[z;\rho,\Im_\rho]=\{b_ez^e+\cdots+b_1z+b_0 \mid b_i\in\mathbb{F}_q \text{ and } e\in\mathbb{N}\}.$$

Then $\mathbb{F}_q[z;\rho,\Im_\rho]$ is a noncommutative ring under the usual addition of 0 polynomials and multiplication defined with respect to $za=\rho(a)z+\Im_\rho(a)$ for all $a\in\mathbb{F}_q$.

Now, consider

$$\mathcal{R}[z;\tau,\delta_\tau]=\{a_lz^l+\cdots+a_1z+a_0 \mid a_i\in\mathcal{R} \text{ and } l\in\mathbb{N}\}$$

where τ is an automorphism of $\mathcal{R}$ and δ_τ is a τ-derivation of $\mathcal{R}$. Then $\mathcal{R}[x;\tau,\delta_\tau]$ is a noncommutative ring under the usual addition of polynomials and multiplication defined with respect to $zr=\tau(r)z+\delta_\tau(r)$ for all $r\in\mathcal{R}$ and extended to all elements of $\mathcal{R}[z;\tau,\delta_\tau]$ by associativity and distributivity. This ring is known as a skew polynomial ring (33).

Example 2.2.2 *Let $f=a_0+a_1z, g=b_0+b_1z+b_2z^2\in\mathcal{R}[z;\tau,\delta_\tau]$. Then*

$$f+g=a_0+b_0+(a_1+b_1)z+b_2z^2=g+f.$$

Also,

$$\begin{aligned}fg&=(a_0+a_1z)(b_0+b_1z+b_2z^2)\\&=a_0(b_0+b_1z+b_2z^2)+a_1z(b_0+b_1z+b_2z^2)\end{aligned}$$

$$
\begin{aligned}
&= a_0b_0 + a_0b_1z + a_0b_2z^2 + a_1\left(\tau(b_0)z + \delta_\tau(b_0)\right) + a_1\left(\tau(b_1)z + \delta_\tau(b_1)\right)z \\
&\quad + a_1\left(\tau(b_2)z + \delta_\tau(b_2)\right)z^2 \\
&= a_0b_0 + a_1\delta_\tau(b_0) + \left(a_0b_1 + a_1\tau(b_0) + a_1\delta_\tau(b_1)\right)z + \left(a_0b_2 + a_1\tau(b_1) + a_1\delta_\tau(b_2)\right)z^2 \\
&\quad + a_1\tau(b_2)z^3;
\end{aligned}
$$

and

$$
\begin{aligned}
gf &= \left(b_0 + b_1z + b_2z^2\right)(a_0 + a_1z) \\
&= b_0(a_0 + a_1z) + b_1z(a_0 + a_1z) + b_2z^2(a_0 + a_1z) \\
&= b_0a_0 + b_0a_1z + b_1\left(\tau(a_0)z + \delta_\tau(a_0)\right) + b_1\left(\tau(a_1)z + \delta_\tau(a_1)\right)z \\
&\quad + b_2\left(\tau^2(a_0)z^2 + 2\tau\left(\delta_\tau(a_0)\right)z + \delta_\tau^2(a_0)\right) + b_2\left(\tau^2(a_1)z^2 + 2\tau\left(\delta_\tau(a_1)\right)z + \delta_\tau^2(a_1)\right)z \\
&= b_0a_0 + b_1\delta_\tau(a_0) + b_2\delta_\tau^2(a_0) + \Big[b_0a_1 + b_1\tau(a_0) + b_1\delta_\tau(a_1) + 2b_2\tau\left(\delta_\tau(a_0)\right) \\
&\quad + b_2\left(\delta_\tau^2(a_1)\right)\Big]z + \Big[b_1\tau(a_1) + b_2\tau^2(a_0) + 2b_2\tau\left(\delta_\tau(a_1)\right)\Big]z^2 + b_2\tau^2(a_1)z^3.
\end{aligned}
$$

Since $fg \neq gf$ and hence $\mathcal{R}[z;\tau,\delta_\tau]$ is a noncommutative polynomial ring.

Definition 2.2.3 *(34; 35) A pseudo-linear transformation $T : \mathcal{R}^n \to \mathcal{R}^n$ is an additive map defined by*

$$T = \tau(v)M + \delta_\tau(v), \tag{2.1}$$

where $v = (v_1, v_2, \ldots, v_n) \in \mathcal{R}^n, \tau(v) = \left(\tau(v_1), \tau(v_2), \ldots, \tau(v_n)\right) \in \mathcal{R}^n, M$ is a matrix of order $n \times n$ over $\mathcal{R}$ and $\delta_\tau(v) = \left(\delta_\tau(v_1), \delta_\tau(v_2), \ldots, \delta_\tau(v_n)\right) \in \mathcal{R}^n$. If $\delta_\tau = 0$, then T is known as a semi-linear transformation.

Throughout this chapter, we fix monic polynomials

$$f(z) = z^n - f_{n-1}z^{n-1} - \cdots - f_1z - f_0 \in R = \mathcal{R}[z;\tau,\delta_\tau], \tag{2.2}$$

$$\mathfrak{f}(z) = z^n - \mathfrak{f}_{n-1}z^{n-1} - \cdots - \mathfrak{f}_1z - \mathfrak{f}_0 \in \mathbb{F}_q[z;\tau,\delta_\tau], \tag{2.3}$$

where $f_{n-1}, \ldots, f_1, f_0 \in \mathcal{R}, \mathfrak{f}_{n-1}, \ldots, \mathfrak{f}_1, \mathfrak{f}_0 \in \mathbb{F}_q$ and $f_0 \neq 0, \mathfrak{f}_0 \neq 0$, and $Rf(z)$ represents the left $\mathcal{R}$-submodule generated by $f(z)$. Again, consider a map $\Omega_f : \mathcal{R}^n \to R / Rf(z)$ defined by $\Omega_f(l) = \left[l(z)\right] \in R / Rf(z)$ for all $l \in \mathcal{R}^n$, where

$$l(z) = \sum_{i=0}^{n-1} l_iz^i,$$

$l_i \in \mathcal{R}$, that is, $\Omega_f(l) = \Omega_f(l_0, l_1, \ldots, l_{n-1}) = \left[l(z)\right] \in R / Rf(z)$ for all $(l_0, l_1, \ldots, l_{n-1}) \in \mathcal{R}^n$. The map Ω_f is an $\mathcal{R}$-linear isomorphism between $\mathcal{R}$-modules.

Now, we recall some definitions for the codes over $\mathcal{R}^n$ as follows.

Definition 2.2.4 (17, (36) *Let* $\mathcal{L} \subseteq \mathcal{R}^n$ *be a non-empty set.*

1. *A code* $\mathcal{L}$ *of length* n *over* $\mathcal{R}$ *is said to be a* (τ, δ_τ)*-linear code if it is a left* R*-submodule of* $R / Rf(z)$*, where* $f(z)$ *is an arbitrary polynomial of degree* n *over* $\mathcal{R}$*. Moreover, if* $f(z)$ *is a central element of* R*, then* $\mathcal{L}$ *is said to be a central* (τ, δ_τ)*-linear code.*
2. *Let* $\mathcal{L}$ *be an* $\mathcal{R}$*-linear code. Then* $\mathcal{L}$ *is said to be a polycyclic code induced by* f *if there exists a vector* $f = (f_0, f_1, \cdots, f_{n-1}) \in \mathcal{R}^n$ *such that for every* $l = (l_0, l_1, \ldots, l_{n-1}) \in \mathcal{L}, (0, l_0, l_1, \ldots, l_{n-2}) + l_{n-1}(f_0, f_1, \ldots, f_{n-1}) \in \mathcal{L}$ *and* f *is called an associated vector of* $\mathcal{L}$*.*

Definition 2.2.5 *1. Let* $\mathcal{L} \subseteq \mathcal{R}^n$ *be an* $\mathcal{R}$*-linear code. Then* $\mathcal{L}$ *is a* τ*-polycyclic code induced by* f *if there exists a vector* $f = (f_0, f_1, \ldots, f_{n-1}) \in \mathcal{R}^n$ *such that for every* $l = (l_0, l_1, \ldots, l_{n-1}) \in \mathcal{L}$*,* $(0, l_0, l_1, \ldots, l_{n-2}) + l_{n-1}(f_0, f_1, \ldots, f_{n-1}) \in \mathcal{L}$*.*

2. $\mathcal{L}$ *is an* (f, τ, δ_τ)*-skew polycyclic code if* $\mathcal{L}$ *is an* $\mathcal{R}$*-linear code such that* $T_f(\mathcal{L}) \subseteq \mathcal{L}$*, where* T_f *is given by Equation (2.1) with*

$$M = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ f_0 & f_1 & \cdots & f_{n-1} \end{pmatrix}$$

and $T_f(\mathcal{L}) = \{T_f(v) \mid v \in \mathcal{L}\}$.

For (f, τ, δ_τ)-skew polycyclic codes there are q^n choices for $f(z) = z^n - f_{n-1}z^{n-1} - \cdots - f_1 z - f_0$ whereas $f(z) = \mathcal{D}(z)$ is the only choice for skew cyclic codes. Now, we consider an example to explain why skew polycyclic codes are more generalized in comparison to skew cyclic codes. If we consider skew cyclic codes of length 4 over $\mathbb{F}_4$, then we need to find the factorization of $z^4 - 1$ in $\mathbb{F}_4[z; \rho]$, where $\rho : \mathbb{F}_4 \to \mathbb{F}_4$ is the Frobenius automorphism defined by $\rho(c) = c^2$. In $\mathbb{F}_4[z; \rho]$, we have

$$\begin{aligned} z^4 - 1 &= \left(z^3 + z^2 + z + 1\right)(z + 1) = \left(wz^3 + z^2 + wz + 1\right)(wz + 1) \\ &= \left(w^2 z^3 + z^2 + w^2 z + 1\right)\left(w^2 z + 1\right) = \left(z^3 + w^2 z^2 + z + w^2\right)(z + w) \\ &= \left(wz^3 + w^2 z^2 + wz + w^2\right)(wz + w) \\ &= \left(w^2 z^3 + w^2 z^2 + w^2 z + w^2\right)\left(w^2 z + w\right) \\ &= \left(z^3 + wz^2 + z + w\right)\left(z + w^2\right) = \left(wz^3 + wz^2 + wz + w\right)\left(wz + w^2\right) \\ &= \left(w^2 z^3 + wz^2 + w^2 z + w\right)\left(w^2 z + w^2\right). \end{aligned}$$

On the other hand, if we consider $(\mathfrak{f}, \rho, \Im_\rho)$-skew polycyclic codes of length 4, then first we have 4^4 choices to construct $\mathfrak{f}(z)$, including $z^4 - 1$ over $\mathbb{F}_4$. Further, we can factorize $\mathfrak{f}(z)$ for each 4^4 choice. One of the choices for $\mathfrak{f}(z)$ is $z^4 + wz^3 + w^2 z^2 + z$, whose factorization in $\mathbb{F}_4[z; \rho, \delta]$, where $\rho : \mathbb{F}_4 \to \mathbb{F}_4$ is the Frobenius automorphism and ρ-derivation

$\Im_\rho : \mathbb{F}_4 \to \mathbb{F}_4$ is defined by $\Im_\rho(c) = \rho(c) - c$, is as follows:

$$\begin{aligned}
z^4 + wz^3 + w^2z^2 + z &= \left(w^2z^2 + w^2z + 1\right)\left(wz^2 + w^2\right) \\
&= \left(z^3 + w^2z^2 + wz + 1\right)z \\
&= z\left(z^3 + w^2z^2 + w^2z = \left(w^2z^2 + w\right)\left(wz^2 + w^2z\right)\right. \\
&= \left(wz^2 + z\right)\left(w^2z^2 + 1\right) = \left(wz^2 + 1\right)\left(w^2z^2 + z\right) \\
&= \left(wz^3 + z\right)wz = (wz + 1)\left(wz^3 + z^2 + z\right) \\
&= \left(z^2 + wz + 1\right)\left(z^2 + w\right) \\
&= \left(w^2z + 1\right)\left(w^2z^3 + wz^2 + wz\right) \\
&= \left(z^2 + w^2\right)\left(z^2 + wz\right) = \left(w^2z^3 + wz^2 + wz + 1\right)w^2z.
\end{aligned}$$

Since every right factor of $\mathfrak{f}(z)$ corresponds an $\left(\mathfrak{f}, \rho, \Im_\rho\right)$-skew polycyclic code of length 4 and having more choices for $\mathfrak{f}(z)$ and their right factors, we must say the class of (f, τ, δ_τ)-skew polycyclic codes is an elegant generalization of skew cyclic codes.

Now, we consider the iterated skew polynomial ring over the ring $\mathcal{R}$. Let $S = \mathcal{R}\left[z_1; \tau_1, \delta_{\tau_1}\right]$ be a skew polynomial ring over the ring $\mathcal{R}$, where τ_1 is an automorphism and δ_{τ_1} is a τ_1-derivation of $\mathcal{R}$. If τ_2 is an endomorphism and δ_{τ_2} is a τ_2-derivation of S, then the skew polynomial ring $B = S\left[z_2; \tau_2, \delta_{\tau_2}\right]$ is called an iterated skew polynomial ring over $\mathcal{R}$. Also, $B = \mathcal{R}\left[z_1; \tau_1, \delta_{\tau_1}\right]\left[z_2; \tau_2, \delta_{\tau_2}\right]$.

For brevity of notation, let $B_i = \mathcal{R}\left[z_1; \tau_1, \delta_{\tau_1}\right]\left[z_2; \tau_2, \delta_{\tau_2}\right]\cdots\left[z_i; \tau_i, \delta_{\tau_i}\right]$ and finite sets $H = \left\{\tau_1, \tau_2, \ldots, \tau_n\right\}$ and $D = \left\{\delta_{\tau_1}, \delta_{\tau_2}, \ldots, \delta_{\tau_n}\right\}$ where τ_i is an endomorphism of B_{i-1} and δ_{τ_i} is the τ_i-derivation of B_{i-1}, for $i = 1, 2, \ldots, n$. Then, by induction, B_i are the iterated skew polynomial rings over $\mathcal{R}, 2 \le i \le n$. If H has only identity automorphism, then B_i is an iterated skew polynomial ring of derivation type, and if D has only zero derivation, then B_i is the iterated skew polynomial ring of automorphism type over $\mathcal{R}$.

Let $\preceq$ be the lexicographical order on $\mathbb{Z} \times \mathbb{Z}$. Now, for any $(c,d), (c',d') \in \mathbb{Z} \times \mathbb{Z}, (c,d) \preceq (c',d')$ if and only if $c < c'$ or $c = c', d \le d'$. Furthermore, if $(c,d) \le (c',d')$ and $(c,d) \ne (c',d')$, then we write $(c,d) \prec (c',d')$ and $\preceq$ is a total order on $\mathbb{Z} \times \mathbb{Z}$. For any polynomial $l(z) \in \mathcal{R}\left[z_1; \tau_1, \delta_{\tau_1}\right]\left[z_2; \tau_2, \delta_{\tau_2}\right]$, we define

$$\mathcal{V}_l = \left\{(a_1, a_2) \mid l(z) \text{ contains a term } mz_1^{a_1}z_2^{a_2}, m \in \mathcal{R}, m \ne 0\right\}.$$

Recall from Sharma and Bhaitwal (27) that the lex-degree of $l(z)$, represented by $lexdeg\left(l(z)\right)$, is the greatest element of $\mathcal{V}_l$ with respect to the total order $\preceq$ on $\mathcal{V}_l$. The lexicographical order $\preceq$ on $\mathcal{R}\left[z_1; \tau_1, \delta_{\tau_1}\right]\left[z_2; \tau_2, \delta_{\tau_2}\right]$ is defined as $l(z) \preceq g(z)$ if and only if $lexdeg\left(l(z)\right) \preceq lexdeg\left(g(z)\right)$, for any $l(z), g(z) \in \mathcal{R}\left[z_1; \tau_1, \delta_{\tau_1}\right]\left[z_2; \tau_2, \delta_{\tau_2}\right]$. The term of the polynomial $l(z)$ corresponding to its lex-degree is said to be the lex-leading term, and its coefficient is known as the lex-leading coefficient.

Now, we present some basic results for iterated skew polynomial rings.

Lemma 2.2.6 *(37, Lemma 3.6) Let R be a ring and δ, a derivation on R. Let $S = R[z;\delta]$ be the skew polynomial ring of derivation type over R and δ_1 be another derivation of R. Then δ_1 can be extended to a derivation of S by $\delta_1(z) = 0$ if and only if δ_1 commutes with δ.*

Theorem 2.2.7 *(37, Theorem 3.7) Let R be a ring and $D = \{\delta_1, \delta_2, \ldots, \delta_n\}$, a finite set of derivations of R. Let B_i be the set of all polynomials in indeterminates $z_1, z_2, \cdots, z_i$ with coefficients in R, for $i = 0,1,2,\cdots,n$, where $B_0 = R$. In B_n, addition is usual and multiplication is defined with respect to $z_i r = r z_i + \delta_i(r)$ for all r in $R, z_i z_j = z_j z_i$ for $i, j = 1,2,\ldots,n$. Then B_i is a skew polynomial ring (of derivation type) over B_{i-1}, for all $i = 1,2,\ldots,n$ if and only if $\delta_i \delta_j = \delta_j \delta_i$, for all $i, j = 1,2,\ldots,n$.*

Definition 2.2.8 *(23) A binary two-dimensional code of area sl is the set of $s \times l$ arrays over $\mathbb{F}_2$, called codewords or code-arrays. A two-dimensional code $\mathcal{L}$ is said to be linear if and only if $\mathcal{L}$ forms a subspace of the sl-dimensional space of the $s \times l$ arrays over $\mathbb{F}_2$. A two-dimensional cyclic code is defined as a two-dimensional linear code such that for each code-array C, all the arrays obtained by permuting the columns or the rows of C cyclically are also code-arrays.*

Now, we fix monic polynomials

$$f(z_1, z_2) = \sum_{a_1=0}^{l} \sum_{a_2=0}^{s} r_{a_1,a_2} z_1^{a_1} z_2^{a_2} \in B = \mathcal{R}[z_1; \tau_1, \delta_{\tau_1}][z_2; \tau_2, \delta_{\tau_2}], \tag{2.4}$$

$$\mathfrak{f}(z_1, z_2) = \sum_{a_1=0}^{l} \sum_{a_2=0}^{s} e_{a_1,a_2} z_1^{a_1} z_2^{a_2} \in B' = \mathbb{F}_q[z_1; \rho_1, \Im_{\rho_1}][z_2; \rho_2, \Im_{\rho_2}] \tag{2.5}$$

where $r_{a_1,a_2} \in \mathcal{R}$ and $e_{a_1,a_2} \in \mathbb{F}_q$.

Define a map $\Gamma_f : \mathcal{R}^n \to B / Bf(z_1, z_2)$ by

$$\Gamma_f(c) = [c(z_1, z_2)] \in B / Bf(z_1, z_2)$$

for all $c \in \mathcal{R}^n$, where $[c(z_1, z_2)] \in B / Bf(z_1, z_2)$ denotes the class of polynomials and

$$c(z_1, z_2) = \sum_{a_1=0}^{l-1} \sum_{a_2=0}^{s-1} c_{a_1,a_2} z_1^{a_1} z_2^{a_2},$$

$c_{a_1,a_2} \in \mathcal{R}^n, n = ls$ and $c = (c_{ij}) \in \mathcal{R}^n$ is an $l \times s$ array. Analogous to classical cyclic codes, we associate the array $c \in \mathcal{R}^n$ to an element $\sum_{a_1=0}^{l-1} \sum_{a_2=0}^{s-1} c_{a_1,a_2} z_1^{a_1} z_2^{a_2}$ in the quotient module $B / Bf(z_1, z_2)$. One can easily check that Γ_f is an $\mathcal{R}$-linear isomorphism.

Definition 2.2.9 *Let $\mathcal{L} \subseteq \mathcal{R}^n$ be a non-empty set and $n = ls$. Then*

1. *Suppose $\mathcal{L}$ is a linear code over $\mathcal{R}$ of length $n = sl$ in which each codeword is viewed as an $l \times s$ array. If $\mathcal{L}$ is closed under row τ_1-shift and column τ_2-shift of codewords, then $\mathcal{L}$ is a two-dimensional skew cyclic code of size $l \times s$ over $\mathcal{R}$ under τ_1 and τ_2.*

2. *If* $\mathcal{L}$ *is an* $\mathcal{R}$*-linear code such that* $T_i(\mathcal{L}) \subseteq \mathcal{L}$*, then* $\mathcal{L}$ *is an* $(M_i, \tau_i, \delta_{\tau_i})$*-skew polycyclic code for* $i = 1,2$*, where* M_1 *is of order* $l \times l, M_2$ *is of order* $s \times s$*, and* $T_i(\mathcal{L}) = \{T_i(v) \mid v \in \mathcal{L}\}$*. Moreover, in the case of a two-dimensional linear code, the pseudo-linear map* T_i *is defined as follows.*

   $T_1(v_{0,a_2}, v_{1,a_2}, \ldots, v_{l-1,a_2}) = \tau_1(v_{0,a_2}, v_{1,a_2}, \ldots, v_{l-1,a_2})M_1 + \delta_{\tau_1}(v_{0,a_2}, v_{1,a_2}, \ldots, v_{l-1,a_2})$, *where* $0 \le a_2 \le s-1$ *and*

$$M_1 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -r_{0,0} & -r_{1,0} & \cdots & -r_{l-1,0} \end{pmatrix}.$$

   $T_2(v_{a_1,0}, v_{a_1,1}, \ldots, v_{a_1,s-1}) = \tau_2(v_{a_1,0}, v_{a_1,1}, \ldots, v_{a_1,s-1})M_2 + \delta_{\tau_2}(v_{a_1,0}, v_{a_1,1}, \ldots, v_{a_1,s-1})$, *where* $0 \le a_1 \le l-1$ *and*

$$M_2 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -r_{0,0} & -r_{0,1} & \cdots & -r_{0,s-1} \end{pmatrix}.$$

3. $\mathcal{L}$ *is an* $(f, \tau_1, \tau_2, \delta_{\tau_1}, \delta_{\tau_2})$*-skew polycyclic code if* $\mathcal{L}$ *is an* $\mathcal{R}$*-linear code such that* $T_{i_f}(\mathcal{L}) \subseteq \mathcal{L}$ *for* $i = 1,2$*, where* T_{i_f} *is as defined above,* $M_1 = (M_1)_f$ *is of order* $l \times l, M_2 = (M_2)_f$ *is of order* $s \times s$ *defined as above and* $T_{i_f}(\mathcal{L}) = \{T_{i_f}(v) \mid v \in \mathcal{L}\}$.
4. *Two* $l \times s$ *arrays* $c = (c_{ij}) \in B/Bf(z_1, z_2)$ *and* $d = (d_{ij}) \in B/Bf(z_1, z_2)$ *are said to be orthogonal if and only if*

$$\sum_{i=0}^{l-1}\sum_{j=0}^{s-1} c_{ij}d_{ij} = 0.$$

Let $\mathcal{L}^{\perp}$ *denotes the dual of* $\mathcal{L}$*. Then a polynomial* $d(z_1, z_2) \in B/Bf(z_1, z_2)$ *is in* $\mathcal{L}^{\perp}$ *if and only if* $c(z_1, z_2)d(z_1, z_2) = 0 \in B/Bf(z_1, z_2)$ *for all codewords* $c(z_1, z_2)$ *of* $\mathcal{L}$*. The dual code of a two-dimensional linear code* $\mathcal{L}$ *is the set of all* $l \times s$ *arrays orthogonal to all codewords of* $\mathcal{L}$*.*

2.3 SKEW GENERALIZED POLYCYCLIC CODES AND THEIR PROPERTIES

In this section, we discuss the algebraic properties of skew generalized polycyclic codes. We generalize skew polycyclic codes to two-dimensional skew polycyclic codes in $B = \mathcal{R}[z_1; \tau_1, \delta_{\tau_1}][z_2; \tau_2, \delta_{\tau_2}]$. The following remarks are the direct consequence of the previous discussion.

Remark 2.3.1 *Let* T_i *be a pseudo-linear transformation on* $\mathcal{R}^n$ *for* $i=1,2$. *If* $p(z_1,z_2)=\sum_{j=0}^{m_1}\sum_{k=0}^{m_2}p_{u,v}z_1^jz_2^k$, *then* $p(T_1,T_2)=\sum_{j=0}^{m_1}\sum_{k=0}^{m_2}p_{u,v}T_1^jT_2^k$ *is not a pseudo-linear transformation.*

Remark 2.3.2 *We have* $\Gamma_f\left(p\left(T_{1_f}T_{2_f}\right)(v)\right)=\left[p(z_1,z_2)v(z_1,z_2)\right]$ *for every* $p(z_1,z_2)\in B$ *and* $v\in\mathcal{R}^n$ *such that* $\Gamma_f(v)=\left[v(z_1,z_2)\right]$. *This implies that the map* Γ_f *is a left* B*-module isomorphism when* $B/Bf(z_1,z_2)$ *is a* B*-module with the product* $p(z_1,z_2)\left[v(z_1,z_2)\right]=\left[p(z_1,z_2)v(z_1,z_2)\right]$ *for any* $p(z_1,z_2),v(z_1,z_2)\in B$.

Based on the previous facts, we have the following characterization of $\left(f,\tau_1,\tau_2,\delta_{\tau_1},\delta_{\tau_2}\right)$-skew polycyclic codes.

Theorem 2.3.3 *Let* $\mathcal{L}$ *be a non-empty subset of* $\mathcal{R}^n$. *Then* $\mathcal{L}$ *is an* $\left(f,\tau_1,\tau_2,\delta_{\tau_1},\delta_{\tau_2}\right)$*-skew polycyclic code if and only if* $\Gamma_f(\mathcal{L})$ *is a left* B*-submodule of* $B/Bf(z_1,z_2)$.

Proof. Suppose $\mathcal{L}$ is a non-empty subset of $\mathcal{R}^n$ and an $\left(f,\tau_1,\tau_2,\delta_{\tau_1},\delta_{\tau_2}\right)$-skew polycyclic code. Then $\mathcal{L}$ is $\mathcal{R}$-linear and $T_{i_f}(\mathcal{L})\subseteq\mathcal{L}$, where T_{i_f} is as given in Definition (2.2.9(3)) and $T_{i_f}(\mathcal{L})=\left\{T_{i_f}(v)\mid v\in\mathcal{L}\right\}$ for $i=1,2$. For $c\in\mathcal{L}$ and $p(z_1,z_2)\in B$, we have

$$\begin{aligned}\Gamma_f\left(p\left(T_{1_f}T_{2_f}\right)(c)\right) &= \left[p(z_1,z_2)c(z_1,z_2)\right]\left(\text{as } \mathcal{L} \text{ is invariant under } T_{i_f}\right)\\ &= p(z_1,z_2)\left[c(z_1,z_2)\right]\left(\text{as } B/Bf(z_1,z_2) \text{ is a } B\text{-module}\right)\\ &= p(z_1,z_2)\Gamma_f(c)\in B/Bf(z_1,z_2).\end{aligned}$$

Therefore, $\Gamma_f(\mathcal{L})$ is a left B-submodule of $B/Bf(z_1,z_2)$.

Conversely, let $\Gamma_f(\mathcal{L})$ be a left B-submodule of $B/Bf(z_1,z_2), c\in\mathcal{L}$ and $r\in\mathcal{R}$. Since $\Gamma_f(\mathcal{L})$ is $\mathcal{R}$-linear, $\Gamma_f(rc)=r\Gamma_f(c)=r\left[c(z_1,z_2)\right]=\left[rc(z_1,z_2)\right]$. This implies that $rc\in\mathcal{L}$ and hence $\mathcal{L}$ is $\mathcal{R}$-linear. Further, from Remark 2.3.2, Γ_f is a left B-module isomorphism, that is, $\Gamma_f\left(p\left(T_{1_f}T_{2_f}\right)(c)\right)=\left[p(z_1,z_2)c(z_1,z_2)\right]$ for every $p(z_1,z_2)\in B$ and $c\in\mathcal{L}$. This shows that $\mathcal{L}$ is invariant under $p\left(T_{1_f},T_{2_f}\right)$, and hence $\mathcal{L}$ is invariant under T_{i_f}. Therefore, $\mathcal{L}$ is an $\left(f,\tau_1,\tau_2,\delta_{\tau_1},\delta_{\tau_2}\right)$-skew polycyclic code.

Now, we introduce the definition of skew generalized polycyclic code in B.

Definition 2.3.4 *Let* $\mathcal{L}\subseteq\mathcal{R}^n$ *be a non-empty set. Then* $\mathcal{L}$ *is said to be an* $\left(f,\tau_1,\tau_2,\delta_{\tau_1},\delta_{\tau_2}\right)$*-skew generalized polycyclic code if* $\Gamma_f(\mathcal{L})=Bg(z_1,z_2)/Bf(z_1,z_2)$, *where* $g(z_1,z_2)\in S\left[z_2;\tau_2,\delta_{\tau_2}\right]$ *is a monic polynomial such that* $f(z_1,z_2)\in Bg(z_1,z_2)$. *The polynomial* $g(z_1,z_2)\in B$ *is said to be the generator polynomial of* $\mathcal{L}$, *and we write* $\mathcal{L}=g(z_1,z_2)_{n,q}^{\tau_1,\tau_2,\delta_{\tau_1},\delta_{\tau_2}}$.

It would be useful if, immediately from the generator polynomial $g(z_1,z_2)$, we could deduce the dimension of the code and also write down a generator matrix. The next theorem allows us to do both together.

Theorem 2.3.5 *Let* $\mathcal{L}=g(z_1,z_2)_{n,q}^{\rho_1,\rho_2,\Im_{\rho_1},\Im_{\rho_2}}$ *be an* $\left(\mathfrak{f},\rho_1,\rho_2,\Im_{\rho_1},\Im_{\rho_2}\right)$*-skew polycyclic code over* $\mathbb{F}_q$ *where* $g(z_1,z_2)=\sum_{a_1=0}^{k_1}\sum_{a_2=0}^{k_2}g_{a_1,a_2}z_1^{a_1}z_2^{a_2}$ *be a monic generating polynomial of* $\mathcal{L}$ *in* $\mathbb{F}_q\left[z_1;\rho_1,\Im_{\rho_1}\right]\left[z_2;\rho_2,\Im_{\rho_2}\right]$ *and* $lexdeg\left(g(z_1,z_2)\right)=(k_1,k_2)$. *Then*

$$\mathcal{B}' = \begin{Bmatrix} g(z_1,z_2), & z_1 g(z_1,z_2), & \cdots & ,z_1^{l-k_1-1} g(z_1,z_2) \\ z_2 g(z_1,z_2), & z_2 z_1 g(z_1,z_2), & \cdots & ,z_2 z_1^{l-k_1-1} g(z_1,z_2) \\ \vdots & \vdots & \cdots & \vdots \\ z_2^{s-k_2-1} g(z_1,z_2), & z_2^{s-k_2-1} z_1 g(z_1,z_2), & \cdots & ,z_2^{s-k_2-1} z_1^{l-k_1-1} g(z_1,z_2) \end{Bmatrix}$$

is a basis for $\mathcal{L}$ *over* $\mathbb{F}_q$ *and* $|\mathcal{L}| = q^{(l-k_1)(s-k_2)}$.

Proof. Let $\mathcal{L} = \langle g(z_1,z_2) \rangle_{n,q}^{\rho_1,\rho_2,\Im_{\rho_1},\Im_{\rho_2}}$ be an $\left(\mathfrak{f},\rho_1,\rho_2,\Im_{\rho_1},\Im_{\rho_2}\right)$-skew polycyclic code over $\mathbb{F}_q$ and $0 \neq p(z_1,z_2) \in \mathcal{L} = g\langle (z_1,z_2) \rangle_{n,q}^{\rho_1,\rho_2,\Im_{\rho_1},\Im_{\rho_2}}$. Then there exists $q(z_1,z_2)$ in B' such that $p(z_1,z_2) = q(z_1,z_2)g(z_1,z_2)$. Also, $lexdeg\left(p(z_1,z_2)\right) \leq (l-1,s-1)$. Hence,

$$\begin{aligned} lexdeg\left(q(z_1,z_2)\right) &= lexdeg\left(p(z_1,z_2)\right) - lexdeg\left(g(z_1,z_2)\right) \\ &\leq (l-1,s-1) - (k_1,k_2) \\ &= (l-k_1-1,s-k_2-1). \end{aligned}$$

Hence, $\mathcal{B}'$ generates $\mathcal{L}$. Next, suppose $\left\{r_{i,j}(z_1,z_2) \mid 0 \leq i \leq l-k_1-1, 0 \leq j \leq s-k_2-1\right\}$ is a subset of $\mathbb{F}_q\left[z_1;\rho_1,\Im_{\rho_1}\right]\left[z_2;\rho_2,\Im_{\rho_2}\right]$ such that

$$\sum_{i=0}^{l-k_1-1} \sum_{j=0}^{s-k_2-1} e_{i,j} z_1^i z_2^j g(z_1,z_2) = 0.$$

If possible, let $r = \sum_{i=0}^{l-k_1-1} \sum_{j=0}^{s-k_2-1} e_{i,j}$ such that $r \neq 0$. Then $lexdeg\, r \leq (l-k_1-1,s-k_2-1)$ and $rg(z_1,z_2) = 0$. Therefore, there exists $t(z_1,z_2) \in \mathbb{F}_q\left[z_1;\rho_1,\Im_{\rho_1}\right]\left[z_2;\rho_2,\Im_{\rho_2}\right]$ such that $rg(z_1,z_2) = 0 = t(z_1,z_2)f(z_1,z_2)$, where $\mathfrak{f} = \sum_{a_1=0}^{l} \sum_{a_2=0}^{s} e_{a_1,a_2} z_1^{a_1} z_2^{a_2}$ and $e_{a_1,a_2} \in \mathbb{F}_q$. Hence,

$$\begin{aligned} lexdeg\left(t(z_1,z_2)f(z_1,z_2)\right) &= lexdeg\left(r(z_1,z_2)\right) + lexdeg\left(g(z_1,z_2)\right) \\ &= (l-k_1-1,s-k_2-1) + (k_1,k_2) \\ &= (l-1,s-1), \end{aligned}$$

which is a contradiction. Hence, $r = \sum_{i=0}^{l-k_1-1} \sum_{j=0}^{s-k_2-1} e_{i,j} = 0$ implies that $e_{i,j} = 0$ for all $0 \leq i \leq l-k_1-1, 0 \leq j \leq s-k_2-1$. Therefore, $\mathcal{B}'$ is a basis for $\mathcal{L}$ over $\mathbb{F}_q$. Thus, $|\mathcal{L}| = q^{(l-k_1)(s-k_2)}$.

Let $\mathcal{L} = g(z_1,z_2)_{n,q}^{\rho_1,\rho_2,\Im_{\rho_1},\Im_{\rho_2}}$ be an $\left(\mathfrak{f},\rho_1,\rho_2,\Im_{\rho_1},\Im_{\rho_2}\right)$-skew generalized polycyclic code over $\mathbb{F}_q$. From the previous theorem, we can conclude that $\mathcal{L}$ is a free left $\mathbb{F}_q$-module of dimension $k = (l-k_1)(s-k_2) = n - lk_2 - sk_1 + k_1k_2$. Now, we define

$$\mathbb{G}_1 = \begin{pmatrix} g \\ T_{1_\mathfrak{f}}(g) \\ \vdots \\ T_{1_\mathfrak{f}}^{l-k_1-1}(g) \end{pmatrix} \tag{2.6}$$

where $g(z_1,z_2) = \sum_{a_1=0}^{k_1} \sum_{a_2=0}^{k_2} g_{a_1,a_2} z_1^{a_1} z_2^{a_2} \in B'$ and $i = 1,2$; that is, g is an $(k_1+1)\times(k_2+1)$ array and we make it $l \times s$ array by post zero-padding in the rows and columns, which is defined as

$$g = \begin{pmatrix} g_{0,0} & g_{0,1} & \cdots & g_{0,k_2} & 0 & \ldots & 0 \\ g_{1,0} & g_{1,1} & \cdots & g_{1,k_2} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_{k_1,0} & g_{k_1,1} & \cdots & g_{k_1,k_2} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

If $\left[g(z_1,z_2)\right] = \Gamma_f(g)$, then the generator matrix of $\mathcal{L}$ is given by

$$\mathbb{G} = \begin{pmatrix} \mathbb{G}_1 \\ T_{2_\mathfrak{f}}(\mathbb{G}_1) \\ \vdots \\ T_{2_\mathfrak{f}}^{s-k_2-1}(\mathbb{G}_1) \end{pmatrix}. \tag{2.7}$$

It is well-known that $dim(\mathcal{L}) + dim\left(\mathcal{L}^\perp\right) = sl$. Therefore, $dim\left(\mathcal{L}^\perp\right) = k' = lk_2 + sk_1 - k_1k_2$. Now, we have the following theorem for the dual code $\mathcal{L}^\perp$ of $\left(\mathfrak{f}, \rho_1, \rho_2, \Im_{\rho_1}, \Im_{\rho_2}\right)$-skew polycyclic code $\mathcal{L} \subseteq \mathbb{F}_q^n$.

Theorem 2.3.6 *Let $\mathfrak{f}(z_1,z_2) \in B'$ and, as in Equation (2.4), $T_{i_\mathfrak{f}}$ be its associated pseudo-linear transformation given in Definition (2.2.9(3)) for $i = 1,2$ and $\mathcal{R} = \mathbb{F}_q$. If $\mathfrak{f}(z_1,z_2) = h(z_1,z_2)g(z_1,z_2) = g(z_1,z_2)h'(z_1,z_2)$ for some monic skew polynomials $g(z_1,z_2), h(z_1,z_2), h'(z_1,z_2) \in B'$ and $e_{0,0} \neq 0$, then the linearly independent columns of the matrix*

$$\mathbb{H} = \begin{pmatrix} \mathbb{H}_1 \\ T_{1_\mathfrak{f}}(\mathbb{H}_1) \\ \vdots \\ T_{1_\mathfrak{f}}^{l-1}(\mathbb{H}_1) \end{pmatrix}$$

form a basis of $\mathcal{L}^{\perp}$ *where*

$$\mathbb{H}_1 = \begin{pmatrix} h' \\ T_{2_\mathfrak{f}}(h') \\ \vdots \\ T_{2_\mathfrak{f}}^{s-1}(h') \end{pmatrix},$$

$$\Gamma_\mathfrak{f}(h') = \left[h'(z_1,z_2)\right], lexdeg\left(h'(z_1,z_2)\right) = (a,b) \text{ and } \mathcal{L} = g(z_1,z_2)_{n,q}^{\rho_1,\rho_2,\Im_{\rho 1},\Im_{\rho 2}}.$$

Proof. Let $a \in \mathcal{L} = \left\langle g(z_1,z_2)\right\rangle_{n,q}^{\rho_1,\rho_2,\Im_{\rho 1},\Im_{\rho 2}}$ and $\Gamma_\mathfrak{f}(h') = \left[h'(z_1,z_2)\right]$. So, $\Gamma_\mathfrak{f}(a) = \left[a(z_1,z_2)\right] = \left[b(z_1,z_2)g(z_1,z_2)\right]$, for some $b(z_1,z_2), a(z_1,z_2) \in B'$. For $i = 1,2$, we have $\Gamma_\mathfrak{f}\left(a\left(T_{1_\mathfrak{f}}, T_{2_\mathfrak{f}}\right)(h')\right) = \left[a(z_1,z_2)h'(z_1,z_2)\right] = \left[\left(b(z_1,z_2)g(z_1,z_2)\right)h'(z_1,z_2)\right] = \left[b(z_1,z_2)\left(g(z_1,z_2)h'(z_1,z_2)\right)\right] = \left[b(z_1,z_2)\mathfrak{f}(z_1,z_2)\right] = [0] \in B'/B'f(z_1,z_2)$. **This implies** $a\left(T_{1_\mathfrak{f}}, T_{2_\mathfrak{f}}\right)(h') = 0$. **Therefore,**

$$0 = a\left(T_{1_\mathfrak{f}}, T_{2_\mathfrak{f}}\right)(h') = \sum_{a_1=0}^{l-1}\sum_{a_2=0}^{s-1} a_{a_1,a_2} T_{1_\mathfrak{f}}^{a_1} T_{2_\mathfrak{f}}^{a_2}(h')$$

where $a(z_1,z_2) = \sum_{a_1=0}^{l-1}\sum_{a_2=0}^{s-1} a_{a_1,a_2} z_1^{a_1} z_2^{a_2}$. This shows that $a\mathbb{H}^t = 0$ for any $a \in \mathcal{L}$ and $\mathbb{H}^t$ is the transpose matrix of $\mathbb{H}$. Further, we note that

$$\Gamma_\mathfrak{f}\left(\left(T_{1_\mathfrak{f}}^e T_{2_\mathfrak{f}}^j\right)(h')\right) = z_1^e z_2^j h'(z_1,z_2) = \begin{cases} z_1^e z_2^j h'(z_1,z_2) & \text{for} 0 \le e \le k_2 - 1, 0 \le j \le l-1 \\ z_1^e z_2^j h'(z_1,z_2) & \text{for } k_2 \le e \le s-1, 0 \le j \le k_1 - 1 \end{cases}$$

where *lexdeg* $\left(h'(z_1,z_2)\right) = (a,b)$. Hence,

$$\begin{Bmatrix} h', & T_{2_\mathfrak{f}}(h'), & \ldots & , T_{2_\mathfrak{f}}^{lk_2-1}(h'), \\ T_{2_\mathfrak{f}}(h'), & T_{2_\mathfrak{f}}^2(h'), & \ldots & , T_{2_\mathfrak{f}}^{s-k_2}(h'), \\ T_{1_\mathfrak{f}}T_{2_\mathfrak{f}}(h'), & T_{1_\mathfrak{f}}T_{2_\mathfrak{f}}^2(h'), & \ldots & , T_{1_\mathfrak{f}}T_{2_\mathfrak{f}}^{s-k_2}(h'), \\ \vdots & \vdots & \vdots & \\ T_{1_\mathfrak{f}}^{k_1-1}T_{2_\mathfrak{f}}(h'), & T_{1_\mathfrak{f}}^{k_1-1}T_{2_\mathfrak{f}}^2(h'), & \ldots & , T_{1_\mathfrak{f}}^{k_1-1}T_{2_\mathfrak{f}}^{s-k_2}(h') \end{Bmatrix}$$

are independent and form a basis for $\mathcal{L}^{\perp}$.

Also, when $\mathcal{R}$ is commutative, the following theorem gives the idea of the dual code $\mathcal{L}^{\perp}$ of $\left(M_i, \tau_i, \delta_{\tau_i}\right)$-skew polycyclic code $\mathcal{L} \subseteq \mathcal{R}^n$ for $i = 1,2$.

Theorem 2.3.7 *Suppose* $\mathcal{R}$ *is commutative and* $\delta_{\tau_i} : \mathcal{R} \to \mathcal{R}$ *is a* τ_i*-derivation, where* $\tau_i \in Aut(\mathcal{R})$ *for* $i = 1,2$. *If* $\mathcal{L} \subseteq \mathcal{R}^n$ *is an* $\left(M_i, \tau_i, \delta_{\tau_i}\right)$*-skew polycyclic code, then* $\mathcal{L}^{\perp}$ *is*

a $\left(\left(M_i^t\right)_{\tau_i^{-1}}, \tau_i^{-1}, \delta''_{\tau_i}\right)$*-skew polycyclic code with* τ_i^{-1}*-derivation* $\delta''_{\tau_i} = -\tau_i^{-1}\delta_{\tau_i}$ *for* $i = 1,2$, *where for a matrix* $E = [(a_{mn})]$, *we denote* $E_{\tau_i^{-1}} = \left[\tau_i^{-1}(a_{mn})\right]$ *and* E^t *its transpose matrix.*

Proof. We define two pseudo-linear transformations T_i and T_i' by $T_i(v) = \tau_i(v)\cdot M_i + \delta_{\tau_i}(v)$ and $T_i'(v) = \tau_i^{-1}(v)\cdot\left(M_i^t\right)_{\tau_i^{-1}} + \delta''_{\tau_i}(v)$ for $i = 1,2$ for every $v \in \mathcal{R}^n$, respectively. For any $a \in \mathcal{L}$ and $b \in \mathcal{L}^{\perp}$, we have $0 = b\cdot T_i^t(a) = b\cdot\left(\tau_i(a)\cdot M_i\right)^t + b\cdot\left(\delta_{\tau_i}(a)\right)^t = b\cdot\left(\tau_i(a)\cdot M_i\right)^t - \delta_{\tau_i}(b)\cdot\left(\tau_i(a)\right)^t$. This implies $0 = b\cdot M_i^t\cdot\left(\tau_i(a)\right)^t - \delta_{\tau_i}(b)\cdot(\tau(a))^t = \left(b\cdot M_i^t - \delta_{\tau_i}(b)\right)\cdot\left(\tau_i(a)\right)^t$; that is, $\left(b\cdot M_i^t - \delta_{\tau_i}(b)\right)\cdot\left(\tau_i(a)\right)^t = 0$, for $i = 1,2$. Therefore, $0 = \tau_i^{-1}(0) = \tau_i^{-1}\left(b\cdot M_i^t - \delta_{\tau_i}(b)\right)\cdot(a)^t = \left(\tau_i^{-1}(b)\cdot\left(M_i^t\right)_{\tau_i^{-1}} - \tau_i^{-1}\delta_{\tau_i}(b)\right)\cdot(a)^t = \left(\tau_i^{-1}(b)\cdot\left(M_i^t\right)_{\tau_i^{-1}} + \delta''_{\tau_i}(b)\right)\cdot(a)^t = T_i'(b)\cdot(a)^t$; that is, $T_i'(b)\cdot(a)^t = 0$ for all $a \in \mathcal{L}, b \in \mathcal{L}^{\perp}$ and $i = 1,2$.

The following examples provide the novelty of our results and the reason we are studying two-dimensional linear codes, especially $\left(\mathfrak{f}, \rho_1, \rho_2, \Im_{\rho_1}, \Im_{\rho_2}\right)$-skew polycyclic codes. Here, all computations are carried out by using Magma software (38).

Example 2.3.8 *Let* $\mathbb{F}_4 = \mathbb{F}_2(w)$ *where* $w^2 + w + 1 = 0$ *and* $B = \mathbb{F}_q\left[z_1; \rho_1, \Im_{\rho_1}\right][z_2; \rho_2, \Im_{\rho_2}]$, *where* $\rho_1, \rho_2 : \mathbb{F}_4 \to \mathbb{F}_4$ *are Frobenius automorphisms defined by* $\rho_1(a) = \rho_2(a) = a^2$, $\Im_{\rho_1} : \mathbb{F}_4 \to \mathbb{F}_4$ *is a* ρ_1*-derivation defined by* $\Im_{\rho_1}(a) = \rho_1(a) - a$ *and* $\Im_{\rho_2}$ *is a zero derivation. Let*

$$f = w^2 z_2^3 + z_1^2 + z_1^2 z_2^2 + w^2 z_1^2 z_2 + w^2 z_2 + 1 = \left(w^2 z_2 + z_1^2 + 1\right)\left(w^2 z_2 + z_2^2 + 1\right)$$

and $g = z_2^2 + w^2 z_2 + 1$. *We have* $l = 2, s = 3, k_1 = 0, k_2 = 2$,

$$g = \begin{pmatrix} 1 & w^2 & 1 \\ 0 & 0 & 0 \end{pmatrix}, M_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and

$$M_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -w^2 & 0 \end{pmatrix}.$$

Also, $n = ls = 6$ *and* $k = (l - k_1)(s - k_2) = (2-0)(3-2) = 2$. *From, Equation (2.7) generator matrix of* $\left(\mathfrak{f}, \rho_1, \rho_2, \Im_{\rho_1}, \Im_{\rho_2}\right)$*-skew generalized polycyclic code* $\mathcal{L}$ *is given by*

$$\mathbb{G} = \begin{pmatrix} g \\ T_{1_\mathfrak{f}}(g) \end{pmatrix} = \begin{pmatrix} 1 & 0 & w^2 & 0 & 1 & 0 \\ 0 & 1 & w^2 & 1 & 0 & 1 \end{pmatrix}.$$

Hence, $\mathcal{L}$ *is a* $\left(\mathfrak{f}, \rho_1, \rho_2, \Im_{\rho_1}, \Im_{\rho_2}\right)$*-skew generalized polycyclic code having parameter [6, 2, 3] which is near optimal (39).*

Example 2.3.9 *Let* $\mathbb{F}_9 = \mathbb{F}_3(w)$, *where* $w^2 + 2w + 2 = 0$ *and* $B = \mathbb{F}_q\left[z_1; \rho_1, \Im_{\rho_1}\right][z_2; \rho_2, \Im_{\rho_2}$, *where* $\rho_1, \rho_2 : \mathbb{F}_9 \to \mathbb{F}_9$ *are Frobenius automorphisms defined by* $\rho_1(a) = \rho_2(a) = a^3, \Im_{\rho_1}$ *and* $\Im_{\rho_2}$ *are zero derivations. Let*

$$\begin{aligned} \mathfrak{f} &= w^3 z_2^3 + w^6 z_2 + w^3 z_2^2 + w^3 + w^6 z_1^3 z_2^2 + w^3 z_1^3 z_2 + z_1^3 \\ &= \left(w^5 z_2 + z_1^3 + w^3\right)\left(w^2 z_2^2 + w z_2 + 1\right) \end{aligned}$$

and $g = w^2 z_2^2 + w z_2 + 1$. *We have* $l = 3, s = 3, k_1 = 0, k_2 = 2$,

$$g = \begin{pmatrix} 1 & w & w^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -w^3 & 0 & 0 \end{pmatrix},$$

and

$$M_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -w^3 & -w^6 & -w^3 \end{pmatrix}.$$

Also, $n = ls = 9$ *and* $k = (l - k_1)(s - k_2) = (3 - 0)(3 - 2) = 3$. *From, Equation (2.7) generator matrix of* $\left(\mathfrak{f}, \rho_1, \rho_2, \Im_{\rho_1}, \Im_{\rho_2}\right)$*-skew generalized polycyclic code* $\mathcal{L}$ *is given by*

$$\mathbb{G} = \begin{pmatrix} g \\ T_{1_\mathfrak{f}}(g) \\ T_{1_\mathfrak{f}}^2(g) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & w & 0 & 0 & w^2 & 0 & 0 \\ 0 & 1 & 0 & 0 & w^3 & 0 & 0 & w^6 & 0 \\ 0 & 0 & 1 & 0 & 0 & w & 0 & 0 & w^2 \end{pmatrix}.$$

Hence, $\mathcal{L}$ *is a* $\left(\mathfrak{f}, \rho_1, \rho_2, \Im_{\rho_1}, \Im_{\rho_2}\right)$*-skew generalized polycyclic code having parameter* [9,3,6], *which is almost MDS.*

2.4 BCH LOWER BOUNDS FOR THE MINIMUM DISTANCE OF SKEW POLYCYCLIC CODES WITH NON-ZERO DERIVATIONS

In this section, we investigate BCH lower bounds for the minimum distance of skew polycyclic codes with non-zero derivations. To do so, first, we derive the series of lemmas useful toward the main result of this section. Following the norm used by Boucher and Ulmer (8), the norm of $i \in \mathbb{N}, N_i^{\rho, \Im_\rho}(b)$ is recursively defined as

$$N_0^{\rho, \Im_\rho}(b) = 1$$

and

$$N_{i+1}^{\rho,\Im_\rho}(b)=\rho\left(N_i^{\rho,\Im_\rho}(b)\right)b+\Im_\rho\left(N_i^{\rho,\Im_\rho}(b)\right)$$

In particular, if $\Im_\rho=0$, we get the classical norm $N_i^{\rho,\Im_\rho}(b)=b\rho(b)\ldots\rho^{i-1}(b)$.

In this section, we derive the BCH lower bounds for the minimum distance of the skew polycyclic codes of the ring $\mathcal{A}=\mathbb{F}_q\left[z;\rho,\Im_\rho\right]$ under certain conditions, where $\Im_\rho$ is a ρ-derivation.

Throughout this section, we consider $\mathfrak{f}(z)=z^n-\mathfrak{f}_{n-1}z^{n-1}-\cdots-\mathfrak{f}_1z-\mathfrak{f}_0$, where $\mathfrak{f}_{n-1},\ldots,\mathfrak{f}_1$, $\mathfrak{f}_0\in\mathbb{F}_q$ and $f_0\neq 0$. Again, consider a map $\Omega_\mathfrak{f}:\mathbb{F}_q^n\to\mathcal{A}/\mathcal{A}\mathfrak{f}(z)$ by $\Omega_\mathfrak{f}(c)=\left[c(z)\right]\in\mathcal{A}/\mathcal{A}\mathfrak{f}(z)$ for all $c\in\mathbb{F}_q^n$ where

$$c(z)=\sum_{j=0}^{n-1}c_jz^j,$$

$c_j\in\mathbb{F}_q$, that is, $\Omega_\mathfrak{f}(c)=\Omega_\mathfrak{f}\left((c_0,c_1,\ldots,c_{n-1})\right)=\left[c(z)\right]\in\mathcal{A}/\mathcal{A}\mathfrak{f}(z)\forall(c_0,c_1,\ldots,c_{n-1})\in\mathbb{F}_q^n$. The map $\Omega_\mathfrak{f}$ is an $\mathbb{F}_q$-linear isomorphism between $\mathbb{F}_q$-modules. Now, we provide the following two lemmas that will be used later in the proof of the main result.

Lemma 2.4.1 *For any non-zero ρ-derivation $\Im_\rho$, in $\mathcal{A}/\mathcal{A}\mathfrak{f}(z)$ we have $P(z)z=1$ and $zQ(z)=1+\Im_\rho\left(Q(z)\right)$, where $P(z)=\mathfrak{f}_0^{-1}z^{n-1}-\mathfrak{f}_0^{-1}\mathfrak{f}_{n-1}z^{n-2}-\cdots-\mathfrak{f}_0^{-1}\mathfrak{f}_2z-\mathfrak{f}_0^{-1}\mathfrak{f}_1$ and $Q(z)=\rho^{-1}\left(\mathfrak{f}_0^{-1}\right)z^{n-1}-\rho^{-1}\left(\mathfrak{f}_0^{-1}\mathfrak{f}_{n-1}\right)z^{n-2}-\cdots-\rho^{-1}\left(\mathfrak{f}_0^{-1}\mathfrak{f}_2\right)z-\rho^{-1}\left(\mathfrak{f}_0^{-1}\mathfrak{f}_1\right)$.*

Proof. Note that in $\mathcal{A}/\mathcal{A}\mathfrak{f}(z)$, we have the following equivalences:

$$z^n-\mathfrak{f}_{n-1}z^{n-1}-\cdots-\mathfrak{f}_1z-\mathfrak{f}_0=0,$$

that is,

$$z^n-\mathfrak{f}_{n-1}z^{n-1}-\cdots-\mathfrak{f}_1z=\mathfrak{f}_0.$$

As $\mathfrak{f}_0\neq 0$,

$$\left(\mathfrak{f}_0\right)^{-1}\left(z^{n-1}-\mathfrak{f}_{n-1}z^{n-2}-\cdots-\mathfrak{f}_1\right)z=1,$$

that is,

$$\left(\mathfrak{f}_0^{-1}z^{n-1}-\mathfrak{f}_0^{-1}\mathfrak{f}_{n-1}z^{n-2}-\cdots-\mathfrak{f}_0^{-1}\mathfrak{f}_2z-\mathfrak{f}_0^{-1}\mathfrak{f}_1\right)z=1.$$

Hence, $P(z)z=1$.

Moreover,

$$z^n-\mathfrak{f}_{n-1}z^{n-1}-\cdots-\mathfrak{f}_1z=\mathfrak{f}_0,$$

that is,

$$\mathfrak{f}_0^{-1}z^n - \mathfrak{f}_0^{-1}\mathfrak{f}_{n-1}z^{n-1} - \cdots - \mathfrak{f}_0^{-1}\mathfrak{f}_2 z^2 - \mathfrak{f}_0^{-1}\mathfrak{f}_1 z = 1$$

This implies $\mathfrak{f}_0^{-1}zz^{n-1} - \mathfrak{f}_0^{-1}\mathfrak{f}_{n-1}zz^{n-2} - \cdots - \mathfrak{f}_0^{-1}\mathfrak{f}_1 z = 1$; that is, $z\rho^{-1}\left(\mathfrak{f}_0^{-1}\right)z^{n-1} - z\rho^{-1}\left(\mathfrak{f}_0^{-1}\mathfrak{f}_{n-1}\right)$ $z^{n-2} - \cdots - z\rho^{-1}\left(\mathfrak{f}_0^{-1}\mathfrak{f}_1\right) = 1 + \Im_\rho\left(\rho^{-1}\left(\mathfrak{f}_0^{-1}\right)\right)z^{n-1} - \Im_\rho\left(\rho^{-1}\left(\mathfrak{f}_0^{-1}\mathfrak{f}_{n-1}\right)\right)z^{n-2} - \cdots - \Im_\rho\left(\rho^{-1}\left(\mathfrak{f}_0^{-1}\mathfrak{f}_1\right)\right)$ This gives $z\left(\rho^{-1}\left(\mathfrak{f}_0^{-1}\right)z^{n-1} - \rho^{-1}\left(\mathfrak{f}_0^{-1}\mathfrak{f}_{n-1}\right)z^{n-2} - \cdots - \rho^{-1}\left(\mathfrak{f}_0^{-1}\mathfrak{f}_1\right)\right) = 1 + \Im_\rho\left(\rho^{-1}\left(\mathfrak{f}_0^{-1}\right)\right)z^{n-1} - \Im_\rho\left(\rho^{-1}\left(\mathfrak{f}_0^{-1}\mathfrak{f}_{n-1}\right)\right)z^{n-2} - \cdots - \Im_\rho\left(\rho^{-1}\left(\mathfrak{f}_0^{-1}\mathfrak{f}_1\right)\right)$.

Hence, $zQ(z) = 1 + \Im_\rho\left(Q(z)\right)$, where $Q(z) = \rho^{-1}\left(\mathfrak{f}_0^{-1}\right)z^{n-1} - \rho^{-1}\left(\mathfrak{f}_0^{-1}\mathfrak{f}_{n-1}\right)z^{n-2} - \cdots - \rho^{-1}\left(\mathfrak{f}_0^{-1}\mathfrak{f}_2\right)z - \rho^{-1}\left(\mathfrak{f}_0^{-1}\mathfrak{f}_1\right)$.

Further, if $\rho = id.$, then $Q(z) = \mathfrak{f}_0^{-1}z^{n-1} - \mathfrak{f}_0^{-1}\mathfrak{f}_{n-1}z^{n-2} - \cdots - \mathfrak{f}_0^{-1}\mathfrak{f}_2 x - \mathfrak{f}_0^{-1}\mathfrak{f}_1 = P(z)$.

Lemma 2.4.2 *Suppose $\mathcal{L} \subseteq \mathbb{F}_q^n$ is a skew polycyclic code, and consider a polynomial $c(z) \in \Omega_{\mathfrak{f}}(\mathcal{L})$ with $w_H(c) = w$. Then there exists $r(z) \in \mathcal{A}$ such that*

$$t(z) = r(z)c(z) = 1 + \sum_{i=1}^{w-1} c_i z^{a_i} + \sum_{i=1}^{w-1} d_i z^{k_i} \in \Omega_{\mathfrak{f}}(\mathcal{L}),$$

where $c_i, d_i \in \mathbb{F}_q^*$ and $a_i, k_i \in \mathbb{N}$ with $a_i, k_i \leq n-1$ for $i = 1,\ldots,w-1$. Also,

$$w_H(t) = w.$$

Proof. Since $w_H(c) = w$, we can write $c(z) = b_{k_0}z^{k_0} + b_{k_1}z^{k_1} + \cdots + b_{k_{w-1}}z^{k_{w-1}}$, where $b_{k_j} \in \mathbb{F}_q, k_0 < \cdots < k_{w-1}$ and $b_{k_j} \neq 0$, for $j = 0,\ldots,w-1$. Hence,

$$\begin{aligned}
c(z) &= b_{k_0}z^{k_0}\left(1 + z^{-k_0}b_{k_0}^{-1}b_{k_1}z^{k_1} + \cdots + z^{-k_0}b_{k_0}^{-1}b_{k_{w-1}}z^{k_{w-1}}\right)\\
&= b_{k_0}z^{k_0}\left(1 + \rho^{-k_0}\left(b_{k_0}^{-1}b_{k_1}\right)z^{k_1-k_0} - \Im_\rho\left(\rho^{-k_0}\left(b_{k_0}^{-1}b_{k_1}\right)\right)z^{k_1} + \cdots\right.\\
&\quad \left. + \rho^{-k_0}\left(b_{k_0}^{-1}b_{k_{w-1}}\right)z^{k_{w-1}-k_0} - \Im_\rho\left(\rho^{-k_0}\left(b_{k_0}^{-1}b_{k_{w-1}}\right)\right)z^{k_{w-1}}\right)\\
&= b_{k_0}z^{k_0}\left(1 + \rho^{-k_0}\left(b_{k_0}^{-1}b_{k_1}\right)z^{k_1-k_0} + \cdots\right.\\
&\quad + \rho^{-k_0}\left(b_{k_0}^{-1}b_{k_{w-1}}\right)z^{k_{w-1}-k_0}\\
&\quad \left. - \Im_\rho\left(\rho^{-k_0}\left(b_{k_0}^{-1}b_{k_1}\right)\right)z^{k_1} - \cdots - \Im_\rho\left(\rho^{-k_0}\left(b_{k_0}^{-1}b_{k_{w-1}}\right)\right)z^{k_{w-1}}\right)\\
z^{-k_0}b_{k_0}^{-1}c(z) &= 1 + \rho^{-k_0}\left(b_{k_0}^{-1}b_{k_1}\right)z^{k_1-k_0} + \cdots + \rho^{-k_0}\left(b_{k_0}^{-1}b_{k_{w-1}}\right)z^{k_{w-1}-k_0}\\
&\quad - \Im_\rho\left(\rho^{-k_0}\left(b_{k_0}^{-1}b_{k_1}\right)\right)z^{k_1} - \cdots - \Im_\rho\left(\rho^{-k_0}\left(b_{k_0}^{-1}b_{k_{w-1}}\right)\right)z^{k_{w-1}}.
\end{aligned}$$

In Lemma 2.4.1, $P(z)z = 1$. This implies $z^{-k_0} = P^{k_0}(z)$.

Therefore,

$$\begin{aligned} P^{k_0}(z)b_{k_0}^{-1}c(z) &= 1+\rho^{-k_0}\left(b_{k_0}^{-1}b_{k_1}\right)z^{k_1-k_0}+\cdots+\rho^{-k_0}\left(b_{k_0}^{-1}b_{k_{w-1}}z^{k_{w-1}-k_0}\right) \\ &\quad -\Im_\rho\left(\rho^{-k_0}\left(b_{k_0}^{-1}b_{k_1}\right)\right)z^{k_1}-\cdots-\Im_\rho\left(\rho^{-k_0}\left(b_{k_0}^{-1}b_{k_{w-1}}\right)\right)z^{k_{w-1}} \\ &\in \Omega_{\mathfrak{f}}(\mathcal{L})\subseteq \mathcal{A}\,/\,\mathcal{A}\mathfrak{f}(z) \end{aligned}$$

Now, put $r(z)=P^{k_0}b_{k_0}^{-1}, c_j=\rho^{-k_0}\left(b_{k_0}^{-1}b_{k_j}\right), d_j=-\Im_\rho\left(\rho^{-k_0}\left(b_{k_0}^{-1}b_{k_j}\right)\right)=-\Im_\rho\left(c_j \text{ and } a_j=k_j-k_0\right.$. This gives

$$t(z)=r(z)c(z)=1+\sum_{i=1}^{w-1}c_iz^{a_i}+\sum_{i=1}^{w-1}d_iz^{k_i}\in\Omega_{\mathfrak{f}}(\mathcal{L}).$$

From the previous equation, we have $w_H(t)=w$.

Inspired by the work done by Hartmann and Tzeng (14), Shi et al. (17), and Cuitiño and Tironi (13), the following result provides the lower bound on the minimum Hamming distance of skew polycyclic codes in $\mathbb{F}_q\left[z;\rho,\Im_\rho\right]$.

Theorem 2.4.3 *Let* $\mathcal{L}=\left\langle g(z)\right\rangle_{n,q}^{\rho,\Im_\rho}$ *be a skew polycyclic code. Suppose there exists* $\beta\in\overline{\mathbb{F}}_q$ *(the algebraic closure of* $\mathbb{F}_q$*) and* $l,m\in\mathbb{Z}_+\cup\{0\}$ *such that* $g\left(\beta_i^{l+mi}\right)=0$, *where* $\beta_i=N_{l+mi}^{\rho,\Im_\rho}(\beta)$ *for* $i=0,\ldots,\Delta-2$. *If* $N_i\left(\beta^m\right)\neq 1, N_{l+mi-1}\left(N_i\left(\beta^m\right)\right)=1$, $N_{l+mi-1}\left(N_i\left(\beta^m\right)\right)=1$ *and* $N_i^{\rho,\Im_\rho}\left(N_{l+mi}^{\rho,\Im_\rho}(\beta)\right)=N_{l+mi}^{\rho,\Im_\rho}\left(N_i^{\rho,\Im_\rho}(\beta)\right)$ *for all* $i=1,\ldots,n-1$, *then* $d_{\mathcal{L}}\geq\Delta$.

Proof. If possible, let there exist a polynomial $c(z)\in\Omega_{\mathfrak{f}}(\mathcal{L})$ with $w_H(c)=w<\Delta$. Then, from Lemma 2.4.2, $c(z)$ can be represented as

$$\begin{aligned} c(z) &= 1+\sum_{i=1}^{w-1}c_i'z^{a_i}+\sum_{i=1}^{w-1}d_iz^{k_i}=1+\sum_{i=1}^{w-1}c_i'z^{a_i}-\sum_{i=1}^{w-1}\Im_\rho\left(c_i'\right)z^{k_i} \\ &= 1+\sum_{i=1}^{w-1}c_i'z^{k_i-k_0}-\sum_{i=1}^{w-1}\Im_\rho\left(c_i'\right)z^{k_i}=1+\sum_{i=1}^{w-1}\left(c_i'z^{-k_0}-\Im_\rho\left(c_i'\right)\right)z^{k_i} \\ &= 1+\sum_{i=1}^{w-1}\left(c_i'P(z)^{k_0}-\Im_\rho\left(c_i'\right)\right)z^{k_i} \\ &= 1+\sum_{i=1}^{w-1}\left(c_i'b_{k_0}-\Im_\rho\left(c_i'\right)\right)z^{k_i}. \end{aligned}$$

Hence, we have

$$c=1+\sum_{i=1}^{w-1}c_iz^{k_i} \tag{2.8}$$

where $c_i',d_i\in\mathbb{F}_q^*$ and $a_i,k_i\in\mathbb{Z}_+$ with $a_i,k_i<n$ and $c_i=c_i'b_{k_0}-\Im_\rho\left(c_i'\right)$, for $i=1,\ldots,w-1$.

Define $L_i=N_{a_i}^{\rho,\Im_\rho}\left(\beta_i\right)$ and $P_j=\sum_{i=1}^{w-1}c_iL_i^j=c\left(\beta_i^j\right)$ and consider a polynomial $p(t)\in\mathbb{F}_q[t]$ given by

$$\begin{aligned} p(t) &= \prod_{i=1}^{w-1}\left(t - L_i^m\right) \\ &= t^{w-1} + p_1 t^{w-2} + \cdots + p_{w-2}t + p_{w-1} \in \mathbb{F}_q[t]. \end{aligned}$$

Then, $L_i^m = N_{a_i}^{\rho,\Im_\rho}\left(\beta_i^m\right) = N_{a_i}^{\rho,\Im_\rho}\left(N_{l+mi}^{\rho,\Im_\rho}\left(\beta^m\right)\right) = N_{l+mi}^{\rho,\Im_\rho}\left(N_{a_i}^{\rho,\Im_\rho}\left(\beta^m\right)\right)$.

If $N_{l+mi}^{\rho,\Im_\rho}\left(N_{a_i}^{\rho,\Im_\rho}\left(\beta^m\right)\right) = 1$, then

$$\begin{aligned} 1 &= N_{l+mi}^{\rho,\Im_\rho}\left(N_{a_i}^{\rho,\Im_\rho}\left(\beta^m\right)\right) \\ &= \rho\left(N_{l+mi-1}^{\rho,\Im_\rho}\left(N_{a_i}^{\rho,\Im_\rho}\left(\beta^m\right)\right)\right)N_{a_i}^{\rho,\Im_\rho}\left(\beta^m\right) + \Im_\rho\left(N_{l+mi-1}^{\rho,\Im_\rho}\left(N_{a_i}^{\rho,\Im_\rho}\left(\beta^m\right)\right)\right) \\ &= \rho(1)N_{a_i}^{\rho,\Im_\rho}\left(\beta^m\right). \end{aligned}$$

This implies $N_{a_i}^{\rho,\Im_\rho}\left(\beta^m\right) = 1$, which contradicts the assumption.

Therefore, $L_i^m = N_{l+mi}^{\rho,\Im_\rho}\left(N_{a_i}^{\rho,\Im_\rho}\left(\beta^m\right)\right) \neq 1$. Hence, 1 is not a root of polynomial $p(t)$, that is, $p(1) \neq 1$.

Next, we have

$$\begin{aligned} 0 &= \sum_{i=1}^{w-1} c_i L_i^l p\left(L_i^m\right) \\ &= \sum_{i=1}^{w-1} c_i L_i^l \left(L_i^{m(w-1)} + p_1 L_i^{m(w-2)} + \cdots + p_{w-2}L_i^m + p_{w-1}\right) \\ &= \sum_{i=1}^{w-1} c_i \left(L_i^{l+m(w-1)} + p_1 L_i^{l+m(w-2)} + \cdots + p_{w-2}L_i^{l+m} + p_{w-1}L_i^l\right) \\ &= \sum_{i=1}^{w-1} c_i L_i^{l+m(w-1)} + p_1 \sum_{i=1}^{w-1} c_i L_i^{l+m(w-2)} + \cdots + p_{w-2}\sum_{i=1}^{w-1} c_i L_i^{l+m} + p_{w-1}\sum_{i=1}^{w-1} c_i L_i^l \\ &= P_{l+m(w-1)} + p_1 P_{l+m(w-2)} + \cdots + p_{w-2}P_{l+m} + p_{w-1}P_l \end{aligned}$$

that is,

$$P_{l+m(w-1)} + p_1 P_{l+m(w-2)} + \cdots + p_{w-2}P_{l+m} + p_{w-1}P_l = 0. \tag{2.9}$$

Since $c(z) \in \Omega_{\mathfrak{f}}(\mathcal{L}) = Rg(z)/\mathcal{A}\mathfrak{f}(z)$, so $c\left(\beta_i^{l+mi}\right) = 0$, for $i = 0,1,\ldots,\Delta-2$. Then $P_{l+mi} = c\left(\beta_i^{l+mi}\right) - 1 = -1$. From Equation (2.9), we have $-1 + p_1(-1) + \cdots + p_{w-2}(-1) + p_{w-1}(-1) = 0$. This implies $p(1) = 0$, which contradicts the assumption. Therefore, $d_{\mathcal{L}} \geq \Delta$.

The following theorem is an extension of Theorem 2.4.3 and gives a better BCH lower bound for the minimum distance of skew polycyclic codes over $\mathbb{F}_q$.

Theorem 2.4.4 *Let $\mathcal{L} = \left\langle g(z)\right\rangle_{n,q}^{\rho,\Im_\rho}$ be a skew polycyclic code. Suppose there exists $\beta \in \overline{\mathbb{F}}_q$ and $l, m_1, m_2 \in \mathbb{Z}_+ \cup \{0\}$ such that $(m_1, m_2) \neq (0,0)$, $g\left(\beta_{i_1}^{l+m_1 i_1 + m_2 i_2}\right) = 0$, where $\beta_{i_1} =$*

$N^{\rho,\Im_\rho}_{l+m_1i_1+m_2i_2}(\beta)$ *for* $i_1=0,1,\ldots,\Delta-2$ *and* $i_2=0,1,\ldots,s$. *If* $N_i\left(\beta^{m_j}\right)\neq 1, N_{l+m_1i_1+m_2i_2-1}\left(N_i\left(\beta^{m_j}\right)\right)=1$ *and* $N^{\rho,\Im_\rho}_i\left(N^{\rho,\Im_\rho}_{l+m_1i_1+m_2i_2}(\beta)\right)=N^{\rho,\Im_\rho}_{l+m_1i_1+m_2i_2}\left(N^{\rho,\Im_\rho}_i(\beta)\right)\forall i=1,2,\ldots,n-1, j=1,2$, then $d_{\mathcal{L}}\geq\Delta+s$.

Proof. From Theorem 2.4.3, we have $d_{\mathcal{L}}\geq\Delta$. If possible, let there exist a polynomial $c(z)\in\Omega_{\mathfrak{f}}(\mathcal{L})$ with $w_H(c)=w$ such that $\Delta\leq w<\Delta+s$. Then from equation (2.8),

$$c(z)=1+\sum_{i=1}^{w-1}c_iz^{a_i}$$

where $c_i\in\mathbb{F}_q^*$ and $a_i\in\mathbb{Z}_+$ with $a_i<n$, for $i=1,\ldots,w-1$.

Now, define $L_i=N^{\rho,\Im_\rho}_{a_i}(\beta_i)$ and $P_j=\sum_{i=1}^{w-1}c_iL_i^j=c\left(\beta_{i_1}^j\right)$ and consider the polynomials $p(t),q(t)\in\mathbb{F}_q[t]$ defined by

$$\begin{aligned}p(t)&=\prod_{i_1=1}^{\Delta-2}\left(t-L_{i_1}^{m_1}\right)\\&=t^{\Delta-2}+p_1t^{\Delta-3}+\cdots+p_{\Delta-3}t+p_{\Delta-2}\in\mathbb{F}_q[t]\\\text{and } q(t)&=\prod_{i_2=1}^{w-1}\left(t-L_{i_2}^{m_2}\right)\\&=t^{w-\Delta+1}+q_1t^{w-\Delta}+\cdots+q_{w-\Delta}t+q_{w-\Delta+1}\in\mathbb{F}_q[t]\end{aligned}$$

Let $r(t)=p(t)q(t)\in\mathbb{F}_q[t]$ and $L_{i_j}^{m_j}=N^{\rho,\Im_\rho}_{a_{i_j}}\left(\beta_{i_1}^{m_j}\right)=N^{\rho,\Im_\rho}_{a_{i_j}}\left(N^{\rho,\Im_\rho}_{l+m_1i_1+m_2i_2}\left(\beta^{m_j}\right)\right)=N^{\rho,\Im_\rho}_{l+m_1i_1+m_2i_2}\left(N^{\rho,\Im_\rho}_{a_{i_j}}\left(\beta^{m_j}\right)\right)$. If $N^{\rho,\Im_\rho}_{l+m_1i_1+m_2i_2}\left(N^{\rho,\Im_\rho}_{a_{i_j}}\left(\beta^{m_j}\right)\right)=1$, then

$$\begin{aligned}1&=N^{\rho,\Im_\rho}_{l+m_1i_1+m_2i_2}\left(N^{\rho,\Im_\rho}_{a_{i_j}}\left(\beta^{m_j}\right)\right)\\&=\rho\left(N^{\rho,\Im_\rho}_{l+m_1i_1+m_2i_2-1}\left(N^{\rho,\Im_\rho}_{a_{i_j}}\left(\beta^{m_j}\right)\right)\right)N^{\rho,\Im_\rho}_{a_{i_j}}\left(\beta^{m_j}\right)+\Im_\rho\left(N^{\rho,\Im_\rho}_{l+m_1i_1+m_2i_2-1}\left(N^{\rho,\Im_\rho}_{a_{i_j}}\left(\beta^{m_j}\right)\right)\right)\\&=\rho(1)N^{\rho,\Im_\rho}_{a_{i_j}}\left(\beta^{m_j}\right).\end{aligned}$$

This implies $N^{\rho,\Im_\rho}_{a_{i_j}}\left(\beta^{m_j}\right)=1$, a contradiction. Therefore, $L_{i_j}^{m_j}\neq 1$, for $j=1,2$. Since $L_{i_1}^{m_1}$ and $L_{i_2}^{m_2}$ are roots of $p(t)$ and $q(t)$, respectively. Hence, 1 is not a root of $r(t)=p(t)q(t)$. Also,

$$0=\sum_{i=1}^{w-1}c_iL_i^lp\left(L_i^{m_1}\right)q\left(L_i^{m_2}\right)$$

$$
\begin{aligned}
&= \sum_{i=1}^{w-1} c_i L_i^l \left(L_i^{m_1(\Delta-2)} + p_1 L_i^{m_1(\Delta-3)} + \cdots + p_{\Delta-2}\right)\left(L_i^{m_2(w-\Delta-1)} + q_1 L_i^{m_2(w-\Delta)} + \cdots + q_{w-\Delta+1}\right)\\
&= \sum_{i=1}^{w-1} c_i \Big[(L_i^{l+m_1(\Delta-2)} + p_1 L_i^{l+m_1(\Delta-3)} + \cdots + p_{\Delta-2} L_i^l)\left(L_i^{m_2(w-\Delta-1)}\right.\\
&\quad \left. + q_1 L_i^{m_2(w-\Delta)} + \cdots + q_{w-\Delta+1}\right)\Big]\\
&= \sum_{i=1}^{w-1} c_i \Big[\left(L_i^{l+m_1(\Delta-2)+m_2(w-\Delta-1)} + p_1 L_i^{l+m_1(\Delta-3)+m_2(w-\Delta-1)} + \cdots + p_{\Delta-2} L_i^{l+m_2(w-\Delta+1)}\right)\\
&\quad + \cdots + q_{w-\Delta+1}\left(L_i^{l+m_1(\Delta-2)} + p_1 L_i^{l+m_1(\Delta-3)} + \cdots + p_{\Delta-2} L_i^l\right)\Big]\\
&= P_{l+m_1(\Delta-2)+m_2(w-\Delta+1)} + p_1 P_{l+m_1(\Delta-3)+m_2(w-\Delta+1)} + \cdots + p_{\Delta-2} P_{l+m_2(w-\Delta+1)}\\
&\quad + \cdots + q_1\left(P_{l+m_1(\Delta-2)+m_2(w-\Delta)} + p_1 P_{l+m_1(\Delta-3)+m_2(w-\Delta)} + \cdots + p_{\Delta-2} P_{l+m_2(w-\Delta)}\right)\\
&\quad + \cdots + q_{w-\Delta+1}\left(P_{l+m_1(\Delta-2)} + p_1 P_{l+m_1(\Delta-3)} + \cdots + p_{\Delta-2} P_l\right),
\end{aligned}
$$

that is,

$$
\begin{aligned}
&P_{l+m_1(\Delta-2)+m_2(w-\Delta+1)} + p_1 P_{l+m_1(\Delta-3)+m_2(w-\Delta+1)} + \cdots + p_{\Delta-2} P_{l+m_2(w-\Delta+1)} + \cdots +\\
&q_1\left(P_{l+m_1(\Delta-2)+m_2(w-\Delta)} + p_1 P_{l+m_1(\Delta-3)+m_2(w-\Delta)} + \cdots + p_{\Delta-2} P_{l+m_2(w-\Delta)}\right) + \cdots +\\
&q_{w-\Delta+1}\left(P_{l+m_1(\Delta-2)} + p_1 P_{l+m_1(\Delta-3)} + \cdots + p_{\Delta-2} P_l\right) = 0.
\end{aligned} \tag{2.10}
$$

Since $c(z) \in \Omega_{\mathrm{f}}(\mathcal{L})$ and $c\left(\beta_{i_1}^{l+m_1 i_1 + m_2 i_2}\right) = 0, P_{l+m_1 i_1 + m_2 i_2} = c\left(\beta_{i_1}^{l+m_1 i_1 + m_2 i_2}\right) - 1 = -1$, for $i_1 = 0,\ldots,\Delta-2, i_2 = 0,\ldots,s$ and $\Delta \le w < \Delta + s$. Hence, from the previous equations, we have $(-1 - p_1 + \cdots + p_{\Delta-3} - p_{\Delta-2})(1 + q_1 + \cdots + q_{w-\Delta} - q_{w-\Delta+1}) = 0$. This implies $r(1) = 0$, which contradicts the assumption. Thus, $w \ge \Delta + s$; that is, $d_{\mathcal{L}} \ge \Delta + s$.

In view of Theorem 2.4.3, we provide an extension of Theorem 2.4.4, which can be proved inductively.

Theorem 2.4.5 *Let* $\mathcal{L} = \langle g(z)\rangle_{n,q}^{\rho,\Im_\rho}$ *be a skew polycyclic code. Suppose that there exist* $\beta \in \overline{\mathbb{F}}_q$ *and* $l, m_1, m_2, \ldots m_r \in \mathbb{Z}_+ \cup \{0\}$ *such that* $(m_1, m_2 \ldots m_r) \ne (0,0), g\left(\beta_{i_1}^{l+m_1 i_1 + \cdots + m_r i_r}\right) = 0$, *where* $\beta_{i_1} = N_{l+m_1 i_1 + \cdots + m_r i_r}^{\rho,\Im_\rho}(\beta)$ *for* $i_1 = 0,\ldots,\Delta-2$ *and* $i_k = 0,\ldots,s_k$ *and* $k = 2,3,\ldots,r$. *If* $N_i\left(\beta^{m_j}\right) \ne 1$, $N_{l+m_1 i_1 + m_2 i_2 + \cdots + m_r i_r - 1}\left(N_i\left(\beta^{m_j}\right)\right) = 1$ *and* $N_i^{\rho,\Im_\rho}\left(N_{l+m_1 i_1 + \cdots + m_r i_r}^{\rho,\Im_\rho}(\beta)\right) = N_{l+m_1 i_1 + \cdots + m_r i_r}^{\rho,\Im_\rho}\left(N_i^{\rho,\Im_\rho}(\beta)\right)$ *for all* $i = 1,\ldots,n-1, j = 1,2,\ldots,r$, *then* $d_{\mathcal{L}} \ge \Delta + \sum_{k=}^{r}$.

Corollary 2.4.6 *Let* $\mathcal{L} = \langle g(z)\rangle_{n,q}^{\rho,\Im_\rho}$ *with* $q \ge n+1$. *If* $\beta \in \overline{\mathbb{F}}_q$ *and* $l, c_1, c_2, \ldots, c_r \in \mathbb{Z}_+ \cup \{0\}$ *such that* $(c_1, c_2, \ldots, c_r) \ne (0,0,\ldots,0), g(z) = lclm\left\{z - \beta^{l+\sum_{k=1}^{r} i_k c_k} : i_1 = 0,\ldots,i_k = 0,\ldots,s_k, \Delta + \sum_{k=2}^{r} s_k = n-k+1\right\}, N_i\left(\beta^{c_j}\right) \ne 1, N_{l+c_1 i_1 + c_2 i_2 + \cdots + c_r i_r}\left(N_i\left(\beta^{c_j}\right)\right) = 1$, *and* $N_i^{\rho,\Im_\rho}\left(N_{l+c_1 i_1 + \cdots + c_r i_r}^{\rho,\Im_\rho}(\beta)\right) = N_{l+c_1 i_1 + \cdots + c_r i_r}^{\rho,\Im_\rho}\left(N_i^{\rho,\Im_\rho}(\beta)\right)$ *for all* $i = 1,\ldots,n-1, j = 1,2,\ldots,r$, *then* $\mathcal{L}$ *is an MDS code.*

Proof. Since $deg\big(g(z)\big) \leq n-k$, the Singleton bound is given by $d_{\mathcal{L}} \leq n - dim(\mathcal{L}) + d_{\mathcal{L}} \leq n - dim(\mathcal{L}) + 1 = n - \big(n - deg\big(g(z)\big)\big) + 1 \leq n - k + 1$. From Theorem 2.4.5, it follows that $d_{\mathcal{L}} \geq n - k + 1$. This implies $d_{\mathcal{L}} = n - k + 1$.

2.5 COMPARATIVE SUMMARY

Table 2.1 provides the MDS codes in $\mathbb{F}_q^n$ for $q = 8, 11$. It can be easily seen that, for some code parameters, cyclic and constacyclic codes do not exist, but one can construct polycyclic codes with the same code parameters. Table 2.2 shows that for some code parameters, polycyclic codes do not exist, but the corresponding skew polycyclic codes exist. Here, all the calculations have been performed using the Magma Software (38).

2.5.1 Comparison of Polycyclic and Skew Polycyclic Codes

It is seen that skew polycyclic codes have better parameters than the polycyclic codes. To make it clear, we consider

$$\mathfrak{f}(z) = z^5 + z^4 + z^3 + wz^2 + 1 \in \mathbb{F}_9[z;\rho]$$

Then we obtain 16 MDS skew polycyclic codes with parameters $[5,2,4]_9$ and $[5,3,3]_9$. The generator polynomials for the codes having parameters $[5,2,4]_9$ are $wz^3 + 2z^2 + w^2z + 1$, $w^2z^3 + w^5z^2 + w^3z + w$, $w^3z^3 + w^6z^2 + 2z + w^2$, $2z^3 + w^7z^2 + w^5z + w^3$, $w^5z^3 + z^2 + w^6z + 2$, $w^6z^3 + wz^2 + w^7z + w^5$, $w^7z^3 + w^2z^2 + z + w^6$, $z^3 + w^3z^2 + wz + w^7$, and codes with parameters $[5,3,3]_9$ are $w^7z^2 + w^2z + 1, z^2 + w^3z + w, wz^2 + 2z + w^2, w^2z^2 + w^5z + w^3, w^3z^2 + w^6z + 2, 2z^2 + w^7z + w^5, w^5z^2 + z + w^6, w^6z^2 + wz + w^7$.

On the other hand, we get only one MDS polycyclic code with parameters $[5,3,3]_9$ and its generator polynomial $z^2 + w^6z + w^7$. There does not exist any polycyclic code with parameters $[5,2,4]_9$.

Example 2.5.1 *Consider* $\mathfrak{f}(z) = z^5 + wz^4 + z^3 + wz^2 + 1 \in \mathbb{F}_8[z;\rho]$. *Here, we obtain 21 MDS skew polycyclic codes with parameters* $[5,4,2]_8$, $[5,1,5]_8$, *and* $[5,3,3]_8$. *If* $\rho = id$, *then we get only two MDS polycyclic codes with parameters* $[5,4,2]_8$ *and* $[5,3,3]_8$ *having corresponding generator polynomials* $z + w^6$ *and* $z^2 + w^3x + w^4$, *respectively.*

Example 2.5.2 *Consider* $\mathfrak{f}(z) = z^5 + z^4 + z^3 + wz^2 + 1 \in \mathbb{F}_9[z;\rho]$. *Here, we obtain 16 MDS skew polycyclic codes with parameters* $[5,4,2]_9$ *and* $[5,3,3]_9$. *If* $\rho = id$, *then we get only one MDS polycyclic code with parameters* $[5,3,3]_9$ *and a generator polynomial* $z^2 + w^6z + w^7$.

Example 2.5.3 *Consider* $\mathfrak{f}(z) = wz^5 + z^4 + z^3 + z^2 + 1 \in \mathbb{F}_8[z;\tau]$. *Here, we obtain 21 MDS skew polycyclic codes with parameters* $[5,2,4]_8$, $[5,4,2]_8$ *and* $[5,1,5]_8$. *If* $\tau = id$, *then we get only two MDS polycyclic codes with parameters* $[5,4,2]_8$ *and* $[5,1,5]_8$ *having corresponding generator polynomials* $z + 4$ *and* $z^4 + 2z^3 + 3z^2 + 4z + 4$, *respectively.*

2.6 CONCLUSION

In this chapter, we have considered skew generalized polycyclic codes over the iterated skew polynomial ring $\mathcal{R}\left[x_1;\tau_1,\delta_{\tau_1}\right]\left[x_2;\tau_2,\delta_{\tau_2}\right]$, which are invariant under pseudo-linear transformation of $\mathcal{R}^n$, where $\mathcal{R}$ is a ring with unity or a finite field $\mathbb{F}_q$. Further, for non-zero derivations, we have studied the BCH lower bounds for the minimum distance of these codes. As an application of our established results, we have also included some examples of MDS codes. Finally, we have provided a comparative summary of our proposed work with polycyclic codes, constacyclic codes, and cyclic codes, which is shown in Tables 2.1 and 2.2.

Table 2.1 MDS Codes in $\mathbb{F}_q^n$ for $q = 8, 11$

q	$n = \deg f$	k	d	g *such that* g *divides* f	a *such that* $f = z^n - a$
8	2	1	2	$z+w^6, z+w^2, z+w^5, z+w, z+w^4, z+1,$ $z+w^3$	$1, w, w^2, w^6$ w^3, w^4, w^5
8	3	2	2	$z+w^6, z+w^2, z+w^5, z+w, z+w^4, z+1,$ $z+w^3$	$1, w, w^2, w^6$ w^3, w^4, w^5
		1	3	$z^2+z+w, z^2+w^3z+1, z^2+wz+w^3$ $z^2+w^4z+w^2, z^2+w^2z+w^5, z^2+w^2z+$ $w,$ $z^2+w^5z+w^4, z^2+w^5z+w, z^2+z+w^4$ $z^2+z+w^2, z^2+w^3z+w^3, z^2+w^6z+w^6$ $z^2+w^3z+w, z^2+w^6z+w^2, z^2+wz+$ w^6 $z^2+w^6+1, z^2+wz+w^4, z^2+w^4z+w^5$ $z^2+w^4z+w^3, z^2+w^2z+w^6, z^2+w^5z+w^5$	$\nexists$
8	4	3	2	$z+w^6, z+w^2, z+w^5, z+w, z+w^4, z+1$ $z+w^3$	$1, w, w^2, w^3$ w^4, w^5, w^6
	4			for codes of the type [4,1,4] and [4,2,3] $\nexists$ $\mathbb{f}$	
8	5	4	2	$z+w^6, z+w^2, z+w^5, z+w, z+w^4, z+1,$ $z+w^3$	$1, w, w^2,$ $w^3, w^4, w^5,$ w^6
	5			for codes of the type [5,3,3],[5,1,5] and [5,2,4] $\nexists$ $\mathbb{f}$	
8	6	5	2	$z+w^6, z+w^2, z+w^5, z+w, z+w^4, z+1,$ $z+w^3$	$1, w, w^2, w^6$ $w^3, w^4, w^5,$
8	6			for codes of the type [6,1,6],[6,4,3],[6,2,5] and [6,3,4] $\nexists$ $\mathbb{f}$	

(*Continued*)

Table 2.1 (Continued) MDS Codes in $\mathbb{F}_q^n$ for $q = 8, 11$

q	$n = \deg f$	k	d	g *such that* g *divides* f	a *such that* $f = z^n - a$
	7			for codes of the type [7,2,6],[7,1,7],[7,5,3],[7,6,2], [7,3,5] and [7,4,4] always $\exists\, f$	1
11	2	1	2	$z+10$	1
11	3	2	2	$z+10$	1
		1	3	z^2+9z+1	$\nexists$
11	4	3	2	$z+10$	1
		2	3	z^2+9z+1	$\nexists$
11	5	4	2	$z+10$	1
		2	4	$z^3+8z^2+3z+10$	$\nexists$
		3	3	z^2+9z+1	$\nexists$
11	6	5	2	$z+10$	1
		2	5	$z^4+7z^3+6z^2+7z+1$	$\nexists$
		3	4	$z^3+8z^2+3z+10$	$\nexists$
		4	3	z^2+9z+1	$\nexists$
11	7	6	2	$z+10$	1
		2	6	$z^5+6z^4+10z^3+z^2+5z+10$	$\nexists$
		3	5	$z^4+7z^3+6z^2+7z+1$	$\nexists$
		4	4	$z^3+8z^2+3z+10$	$\nexists$
		5	3	z^2+9z+1	$\nexists$
11	8	7	2	$z+10$	1
		3	6	$z^5+6z^4+10z^3+z^2+5z+10$	$\nexists$
		4	5	$z^4+7z^3+6z^2+7z+1$	$\nexists$
		5	4	$z^3+8z^2+3z+10$	$\nexists$
11	8	2	7	$z^6+5z^5+4z^4+2z^3+4z^2+5z+1$	$\nexists$
		6	3	z^2+9z+1	$\nexists$
11	9	8	2	$z+10$	1
		4	6	$z^5+6z^4+10z^3+z^2+5z+10$	$\nexists$
		5	5	$z^4+7z^3+6z^2+7z+1$	$\nexists$
		6	4	$z^3+8z^2+3z+10$	$\nexists$

Table 2.1 (Continued) MDS Codes in $\mathbb{F}_q^n$ for $q = 8, 11$

q	$n = \deg f$	k	d	g *such that* g *divides* f	a *such that* $f = z^n - a$
		3	7	$z^6 + 5z^5 + 4z^4 + 2z^3 + 4z^2 + 5z + 1$	∄
		7	3	$z^2 + 9z + 1$	∄
		2	8	$z^7 + 4z^6 + 10z^5 + 9z^4 + 2z^3 + z^2 + 7z + 10$	∄
11	10	9	2	$z + 10$	1
		5	6	$z^5 + 6z^4 + 10z^3 + z^2 + 5z + 10$	∄
		6	5	$z^4 + 7z^3 + 6z^2 + 7z + 1$	∄
		7	4	$z^3 + 8z^2 + 3z + 10$	∄
		4	7	$z^6 + 5z^5 + 4z^4 + 2z^3 + 4z^2 + 5z + 1$	∄
		8	3	$z^2 + 9z + 1$	∄
		3	8	$z^7 + 4z^6 + 10z^5 + 9z^4 + 2z^3 + z^2 + 7z + 10$	∄
		2	9	$z^8 + 3z^7 + 6z^6 + 10z^5 + 4z^4 + 10z^3 + 6z^2 + 3z + 1$	∄

Table 2.2 Comparison between the Number of MDS Skew Polycyclic Codes and Polycyclic Codes

q	n	$f(z)$	d	*Parameters of polycyclic codes*	*No. of MDS polycyclic*	*Parameters of skew polycyclic codes*	*No. of MDS skew polycyclic codes*
8	8	$wz^8 + w^2z^7 + wz^6 + wz^4$	3		∄	$[8,6,3]_8$	7
8	9	$z^9 + z^7 + wz^6 + wz^5 + w^2z^4 + wz^3 + w^2z^2 + w$	2,3	$[9,8,2]_8$	1	$[9,8,2]_8$ $[9,7,3]_8$	14
8	9	$z^9 + wz^5 + z^4 + z^2 + 1$	2,3	$[9,8,2]_8$	1	$[9,8,2]_8$	21
9	8	$wz^8 + w^2z^7 + wz^6 + wz^4 + z^2 + 1$	2,3		∄	$[8,7,2]_9$ $[8,6,3]_9$	23
9	5	$wz^5 + z^4 + z^3 + z^2 + 1$	2,3,4,5	$[5,4,2]_9$	1	$[5,4,2]_9$ $[5,3,3]_9$ $[5,2,4]_9$ $[5,1,5]_9$ $[6,5,2]_9$ $[6,4,3]_9$ $[6,3,4]_9$	24
9	6	$wz^6 + wz^4 + z^3 + wz^2 + 1$	2,3,4	$[6,5,2]_9$	2		

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Chapter 3

Construction of Block-Matrix LCD Codes

Ashwini Kumar, R. S. Raja Durai, and Meenakshi Devi

3.1 INTRODUCTION

A $\mathcal{C}(n,k,d)$ linear code of length n is a k-dimensional subspace of $\mathbb{F}_{q^n}$ over the base field $\mathbb{F}_q$ with minimum distance d. A linear code $\mathcal{C}$, together with its dual $\mathcal{C}^{\perp}$, having trivial intersection $\mathcal{C} \cap \mathcal{C}^{\perp} = \{0\}$, is termed *linear code with a complementary dual* (abbreviated LCD). This special class of linear codes was introduced by Massey (1992). In addition to being asymptotically good (Massey 1992), LCD codes attracted researchers due to their recent application in the study of information protection against the so-called attacks, namely, *fault injection attacks* studied in Bringer et al. (2014) and *side-channel attacks* (SCAs) studied in Carlet and Guilley (2016, 2018) by means of orthogonal direct sum masking of information in $\mathbb{F}_2^n = \mathcal{C}_1 \oplus \mathcal{C}_2$ for some linear codes $\mathcal{C}_1$ and $\mathcal{C}_2$ of length n. Carlet and Guilley (2016) showed that binary LCD codes can be obtained from non-binary LCD codes in characteristic 2. Due to their distinct properties and applications in theory and practice, researchers found interest in the constructions and characterization of LCD codes. Consequently, several constructions of codes having complementary dual has been considered by researchers like Galvez et al. (2018), Krisnawati and Nisa (2017), and Li (2017). Using the concatenation technique, which involves employing inner and outer codes, constructions of LCD codes was given by Carlet and Guilley (2016), Carlet et al. (2018), and Li (2017).

Introduced by Gabidulin (1985), the class of rank-distance codes is defined as a subset of an n-dimensional space $\mathbb{F}_{q^N}^n$ of n-vectors over an extension field $\mathbb{F}_{q^N}$, where $n \leq N$. As the construction of MRD codes in Gabidulin (1985) was based on the requirement of $n <= N$ basis elements from $\mathbb{F}_{q^N}$, the procedure cannot be used to define codes of length longer than N. Constructions of MRD codes for $n > N$ in $\mathbb{F}_{q^N}$ are unseen until the discovery of reducible rank codes in Gabidulin et al. (2003), wherein reducible rank codes were in fact introduced for the purpose of employing them in a public-key cryptosystem scheme. Further, constructions of codes of length $n > N$ based on the Cartesian product of several MRD codes was done by Shi and Huang (2018) & Kumar and Durai (2021).

Motivated by this, the objective of the chapter is to construct a new class of MRD codes of length n in $\mathbb{F}_{q^N}$ for $n > N$. A class of MRD code $\mathcal{C}'$ of length $n > N$ is obtained as a special class of reducible rank codes by considering the constituent code $\mathcal{C}$ as an MRD code. In doing so, the construction invariably requires the use of LCD codes derived from MRD codes. Further, the constructed class of codes is shown to be LCD codes. These constructed codes are in fact LCD MRD codes by Kumar and Durai (2021): linear codes $\mathcal{C}'(n',k',d)$ of length $n' \geq n$

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and dimension $k' \geq k$ constructed from MRD codes $\mathcal{C}(n,k,d)$ satisfying $\mathcal{C} \cap \mathcal{C}^{\perp} = \{\mathbf{0}\}$ are termed LCD MRD codes.

Recently a general construction of a class of Cartesian product (LCD MRD) codes of length $n > N$ is constructed from the Cartesian product of ℓ generic LCD MRD codes $\mathcal{C}_1(n,k_1), \mathcal{C}_2(n,k_2), \ldots, \mathcal{C}_\ell(n,k_\ell)$ (Kumar and Durai 2021). The LCD codes are characterized as trivial and non-trivial LCD codes therein. Also a construction of Block-Matrix codes of length $n > N$ using a *self-dual* basis of fields with characteristic 2 is proposed to be LCD. Motivated by this to further investigate the structure as well as the properties of Block-Matrix codes, the present work is devoted to various constructions of Block-Matrix LCD codes.

Organization of the Chapter

The rest of the chapter is organized as: Section 3.2 enlighens Preliminaries and Notations. Section 3.3 introduces a general class of Block-Matrix LCD codes constructed using ℓ *self-dual* bases of the extension fields $\mathbb{F}_{\left(2^t\right)^{2k}}$ with characteristic 2 for any $t > 0$. A special case in which the Block-Matrix LCD codes are constructed using a single *self-dual* basis is also discussed. An approach to determine the parity-check matrices of the constructed Block-Matrix LCD codes is given in Section 3.4. Further, construction of Block-Matrix LCD codes over the field $\mathbb{F}_{(2^t)^{2k+1}}$ is presented in Section 3.5. Providing a simple application, we demonstrate the encoding and decoding procedure for the noiseless case in Section 3.6. Several illustrations in support of the proposed constructions are given in Section 3.7. And, finally, section 3.8 concludes the chapter with future scope.

3.2 PRELIMINARIES AND NOTATIONS

Some important concepts and previously obtained results are introduced in what follows.

3.2.1 Finite Fields

Consider an ordered basis $\mathbf{g} = (g_1, g_2, \ldots, g_n)$ of $\mathbb{F}_{q^n}$ over the base field $\mathbb{F}_q$. Define a trace function of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ as

$$\mathbf{Tr}(\gamma) = \sum_{i=0}^{n-1} \gamma^{q^i}$$

for arbitrary element γ of $\mathbb{F}_{q^n}$. Then a basis $\mathbf{g}^* = \left(g_1^*, g_2^*, \ldots, g_n^*\right)$ of $\mathbf{g}$ satisfying $\mathbf{Tr}\left(g_i g_j^*\right) = \delta_{i,j}$ is called the dual basis of $\mathbf{g}$, where $\delta_{i,j}\,(1 \leq i, j \leq n)$ is the Kronecker delta function. The Gram matrix $\mathbf{T}_m = \left[\mathbf{Tr}\left(g_i g_j\right)\right]_{i,j=1}^{m}$ is a base change matrix from $\mathbf{g}$ to $\mathbf{g}^*$. It is worth mentioning that $\mathbf{T}_m$ is the identity matrix $\mathbf{I}_m$ of order m, provided $\mathbf{g}$ is a *self-dual* basis. If $\mathbf{g}^* = \mathbf{g}$, then the basis $\mathbf{g} = (g_1, g_2, \ldots, g_n)$ is called a *self-dual* basis, and it happens when $\mathbf{Tr}\left(g_i g_j\right) = \delta_{i,j}$. A basis $\mathbf{g} = (g_1, g_2, \ldots, g_n)$ is called *trace-orthogonal* when $\mathbf{Tr}\left(g_i g_j\right) = 0$ for $i \neq j$. The *trace-orthogonal* basis is said to be *self-dual* (*self-complementary*) if $\mathbf{Tr}\left(g_i g_j\right) = 1$ for $i = j$. It is known that not all extension fields contain *self-dual* bases. The extension field $\mathbb{F}_{q^n}$ contains a *self-dual* basis if and only if either q is even or both q and n are odd (Seroussi and Lempel 1980). The *self-dual* bases play an important role in all the constructions of this chapter.

3.2.2 Rank Metric Codes with Complementary Duals

Rank metric codes are defined over an extension field $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$. A rank-distance code $\mathcal{C}(n,k,d)$ of length n and dimension k over $\mathbb{F}_{q^m}$ is defined as a k-dimensional subspace of the normed vector space $\mathbb{F}_{q^m}^n$ over $\mathbb{F}_{q^m}, n \leq m$. In practice, messages are transmitted as vectors $\mathbf{x} \in \mathcal{C}$. Due to the noise in the channel, errors are added to $\mathbf{x}$. The errors are nothing but the elements of $\mathbb{F}_{q^m}^n$ and are quantified by the rank norm; $\text{rank}(\mathbf{v})$ defined on $\mathbb{F}_{q^m}^n$, where $\text{rank}(\mathbf{v})$ is the maximal number of linearly independent coordinates of $\mathbf{v} \in \mathbb{F}_{q^m}^n$ over the base field $\mathbb{F}_q$. This rank norm induces the rank metric $\mathbf{d}_R(\mathbf{v}_1, \mathbf{v}_2) = \text{rank}(\mathbf{v}_1 - \mathbf{v}_2)$ on the space $\mathbb{F}_{q^m}^n$ with respect to the extension field $\mathbb{F}_{q^m}$ so that the minimum distance of $\mathcal{C}$ is $d = \min\{\mathrm{d}_R(\mathbf{v}_1, \mathbf{v}_2) | \mathbf{v}_1 \neq \mathbf{v}_2$ for $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{C}\}$. Due to its importance in network coding given in (Silva et al. 2007), the rank metric has attracted considerable attention recently. In addition to this, enhanced work on the properties of rank metric codes was proposed by Loidreau (2008).

The rank-distance codes which attain the singleton-like bound $d = n - k + 1$ are called *maximum rank-distance* (MRD) codes. The generator matrix for an $\mathcal{C}(n,k,d)$ MRD code over $\mathbb{F}_{q^m}$ is defined as

$$\mathbf{G} = \begin{pmatrix} g_1 & g_2 & \cdots & g_m \\ g_1^q & g_2^q & \cdots & g_m^q \\ \vdots & \vdots & \ddots & \vdots \\ g_1^{q^{k-1}} & g_2^{q^{k-1}} & \cdots & g_m^{q^{k-1}} \end{pmatrix},$$

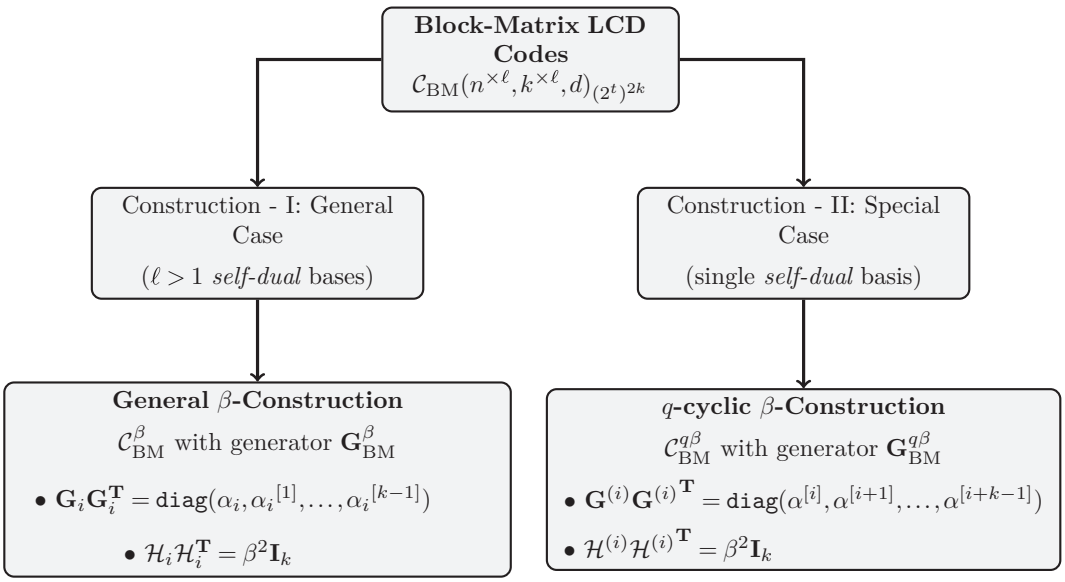


Figure 3.1 Construction of Block-Matrix LCD codes.

where $g_1, g_2, \ldots, g_m \in \mathbb{F}_{q^m}$ are linearly independent over $\mathbb{F}_q$. Considering $n = m$, in (Raja Durai 2003) and (Kandasamy et al. 2012), a certain class of MRD codes having complementary duals was identified. LCD codes constructed using a *trace-orthogonal* (resp. *self-dual*) basis over $\mathbb{F}_{2^n}$ (resp. $\mathbb{F}_{q^n}$) from the class of MRD codes defined over $\mathbb{F}_{2^n}$ in (Raja Durai 2003) (resp. $\mathbb{F}_{q^n}$ in Liu and Liu 2019) are known as LCD MRD codes. All the constructions of LCD MRD codes using a *self-dual* basis by means of a generator matrix $\mathbf{G}$ were found to satisfy the condition $\mathbf{G}\mathbf{G}^{\mathrm{T}} = \mathbf{I}_k$**!** However, recently in (Kumar and Durai 2021), the LCD MRD codes are classified as *trivial* and *non-trivial* LCD MRD codes.

Theorem 3.2.1 Kumar and Durai (2021) *Let q be even or both q and n be odd. Suppose* $\mathbf{h}$ *is a self-dual basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. An MRD code $\mathcal{C}$ defined either by* $\mathbf{H} = \left\langle \mathbf{h}^{[0]} \right\rangle_{n-k}$ *or* $\mathbf{G} = \left\langle c\mathbf{h}^{[n-k]} \right\rangle_k$ *is an* LCD MRD *code with* $\mathbf{G}\mathbf{G}^{\mathrm{T}} = \mathrm{diag}\left(\left(c^2\right)^{[0]}, \left(c^2\right)^{[1]}, \ldots, \left(c^2\right)^{[k-1]}\right)$ *for some* $c \in \mathbb{F}_{q^n}^*$.

The aforementioned result classifies an LCD MRD code $\mathcal{C}$ in a general way as (i) a *trivial* LCD MRD code for $c = 1\left(\mathbf{G}\mathbf{G}^{\mathrm{T}} = \mathbf{I}_k\right)$ and (ii) a *non-trivial* LCD MRD code for $c \neq 1$ $(\mathbf{G}\mathbf{G}^{\mathrm{T}} \neq \mathbf{I}_k)$. Associated with each LCD MRD code $\mathcal{C}$ is a mapping known as an *orthogonal projector* $\Pi_{\mathcal{C}} : \mathbb{F}_{q^n}^n \mapsto \mathcal{C}$ defined as $\Pi_{\mathcal{C}} = \mathbf{G}\left(\mathbf{G}^{\mathrm{T}}\mathbf{G}\right)^{-1}\mathbf{G}^{\mathrm{T}}$. In (Massey (1992) a necessary and sufficient condition for a code $\mathcal{C}$ defined by $\mathbf{G}$ to be LCD is given as $\mathbf{G}\mathbf{G}^{\mathrm{T}}$ must be non-singular. If $\mathcal{C}$ is LCD then so is $\mathcal{C}^{\perp}$.

Remark 3.2.1 *The generator-parity relation obtained for an MRD code is based only on a self-dual basis of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$. In this way, the generator matrix of an MRD code is uniquely determined up to a scalar multiple. As a special case, when $c \in \mathbb{F}_p^*$, that is, $c \in \mathbb{F}_p^* \subset \mathbb{F}_q^*\left(q = p^n\right)$,* $\mathbf{G}\mathbf{G}^{\mathrm{T}} = \mathrm{diag}\left(c^2, \left(c^2\right)^{q^1}, \ldots, \left(c^2\right)^{q^{k-1}}\right) = \mathrm{diag}\left(c^2, c^2, \ldots, c^2\right)$ *is a scalar matrix with* $\left|\mathbf{G}\mathbf{G}^{\mathrm{T}}\right| = \left(c^2\right)^k \in \mathbb{F}_p^*$.

Remark 3.2.2 *It is worth mentioning that while the trace-orthogonal matrix of a self-dual basis* $\mathbf{h}$ *is an identity matrix, the trace-orthogonal matrix of the basis* $\mathbf{g} = c\mathbf{h}^{[n-k]}$ *is not an identity matrix! The condition* $\mathbf{G}\mathbf{G}^{\mathrm{T}} = \mathbf{I}_k$ *is then only a particular case when* $c = 1$.

The construction of MRD codes by Gabidulin (1985) required $n \leq m$ basis elements from $\mathbb{F}_{q^m}$; however, the construction of MRD codes of length longer than m over $\mathbb{F}_{q^m}$ was not discussed. Until 2003, the construction of MRD codes for $n > m$ in $\mathbb{F}_{q^m}$ is unseen. In Gabidulin et al. (2003), a class of *reducible rank* codes was introduced along with their applicability in a public-key cryptosystem. This family of codes includes MRD codes of lengths $n > m$. To facilitate protection against information leakage, an ℓ-fold Cartesian product construction of MRD codes of length greater than m was proposed by Silva and Kschischang (2008). Constructing LCD codes of length $n > m$ from MRD codes is of significance in theory and practice.

3.2.3 Symbols and Notations

Symbols and notations used throughout this work are briefed in what follows.

- Denote the q-power q^i by $[i]$, where the exponent in the q-power is calculated modulo m; $[i] = q^{i \bmod m}$, where m is the degree of extension of the finite field $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$.

- For $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_{q^m}^n$, define the s-power shift of $\mathbf{x}$ as the vector $\left(x_1^{[s]}, x_2^{[s]}, \ldots, x_n^{[s]}\right)$. By abuse of notation, let the generator matrix $\mathbf{G}$ of an MRD code with the ki-power shifts of $\mathbf{g}, i = 0, 1, 2, \ldots, k-1$ be conveniently abbreviated as $\mathbf{G} = \langle \mathbf{g} \rangle_k$.
- An $r \times r$ identity matrix is denoted as $\mathbf{I}_r$ or simply $\mathbf{I}$. An all-zero matrix of order $r \times t$ is denoted by $\mathbf{0}_{r \times t}$; while $\mathbf{0}_r$ represents a square matrix of order r, an all-zero vector is denoted by $\mathbf{0}$.

3.3 SELF-DUAL CONSTRUCTIONS OF BLOCK-MATRIX LCD CODES

In Kumar and Durai (2021), a *self-dual* β-construction of a class of Block-Matrix codes over $\mathbb{F}_{(2^n)^n}$ of length $n\ell$ was proposed through a single *self-dual* basis by considering the generator and parity-check matrices of (even number of) ℓ constituent codes. In that, it is shown that the constructed codes of length greater than n are LCD codes when the constituent codes are *non-trivial* LCD MRD codes and their duals are *trivial* LCD MRD codes. Extending the construction of Block-Matrix LCD codes of Kumar and Durai (2021), this section presents two generalized constructions of Block-Matrix LCD codes over the field $\mathbb{F}_{\left(2^t\right)^n}$ of characteristics 2 with $t > 0$, each employing (odd or even number of) $\ell > 1$ constituent LCD MRD codes. While the first construction makes use of at most ℓ *self-dual* bases, the second construction utilizes a single *self-dual* basis alone to define the ℓ LCD MRD constituent codes and their associated (trivial/non-trivial) duals.

3.3.1 Construction-I: General β-Construction

Let $\mathbf{h}_1 = \{\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1n}\}, \mathbf{h}_2 = \{\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2n}\}, \ldots, \mathbf{h}_\ell = \{\alpha_{\ell 1}, \alpha_{\ell 2}, \ldots, \alpha_{\ell n}\}$ be ℓ *self-dual bases* of $\mathbb{F}_{2^t}^n$ over $\mathbb{F}_{2^t}$ that are not necessarily distinct, where and hereafter $n = 2k$. Consider k-dimensional LCD MRD codes $\mathcal{C}_1(n,k,d), \mathcal{C}_2(n,k,d), \ldots, \mathcal{C}_\ell(n,k,d)$ over $\mathbb{F}_{(2^t)^{2k}}$ with the respective $k \times n$ parity-check matrices $\mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_\ell$:

$$\mathbf{H}_1 = \begin{pmatrix} \mathbf{h}_1^{[0]} \\ \mathbf{h}_1^{[1]} \\ \vdots \\ \mathbf{h}_1^{[k-1]} \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{11}^{[1]} & \alpha_{12}^{[1]} & \cdots & \alpha_{1n}^{[1]} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{11}^{[k-1]} & \alpha_{12}^{[k-1]} & \cdots & \alpha_{1n}^{[k-1]} \end{pmatrix},$$

$$\mathbf{H}_2 = \begin{pmatrix} \mathbf{h}_2^{[0]} \\ \mathbf{h}_2^{[1]} \\ \vdots \\ \mathbf{h}_2^{[k-1]} \end{pmatrix} = \begin{pmatrix} \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \alpha_{21}^{[1]} & \alpha_{22}^{[1]} & \cdots & \alpha_{2n}^{[1]} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{21}^{[k-1]} & \alpha_{22}^{[k-1]} & \cdots & \alpha_{2n}^{[k-1]} \end{pmatrix},$$

$$\vdots$$

$$\mathbf{H}_\ell = \begin{pmatrix} \mathbf{h}_\ell^{[0]} \\ \mathbf{h}_\ell^{[1]} \\ \vdots \\ \mathbf{h}_\ell^{[k-1]} \end{pmatrix} = \begin{pmatrix} \alpha_{\ell 1} & \alpha_{\ell 2} & \cdots & \alpha_{\ell n} \\ \alpha_{\ell 1}^{[1]} & \alpha_{\ell 2}^{[1]} & \cdots & \alpha_{\ell n}^{[1]} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{\ell 1}^{[k-1]} & \alpha_{\ell 2}^{[k-1]} & \cdots & \alpha_{\ell n}^{[k-1]} \end{pmatrix}$$

and the associated $k \times n$ generator matrices $\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_\ell$:

$$\mathbf{G}_1 = \begin{pmatrix} \mathbf{g}_1^{[0]} \\ \mathbf{g}_1^{[1]} \\ \vdots \\ \mathbf{g}_1^{[k-1]} \end{pmatrix}, \mathbf{G}_2 = \begin{pmatrix} \mathbf{g}_2^{[0]} \\ \mathbf{g}_2^{[1]} \\ \vdots \\ \mathbf{g}_2^{[k-1]} \end{pmatrix}, \ldots, \mathbf{G}_\ell = \begin{pmatrix} \mathbf{g}_\ell^{[0]} \\ \mathbf{g}_\ell^{[1]} \\ \vdots \\ \mathbf{g}_\ell^{[k-1]} \end{pmatrix},$$

where $\mathbf{g}_i^{[0]} = c_i \mathbf{h}_i^{[n-k]}$ for some $c_i \in \mathbb{F}_{\left(2^t\right)^{2k}}$. Then each $\mathbf{G}_i \mathbf{G}_i^{\mathbf{T}}$ is a $k \times k$ diagonal matrix of the form $\operatorname{diag}\left(\alpha_i, \alpha_i^{[1]}, \ldots, \alpha_i^{[k-1]}\right)$ and $\mathbf{H}_i \mathbf{H}_i^{\mathbf{T}} = \mathbf{I}_k$ Kumar and Durai (2021), where and hereafter $\alpha_i = c_i^2$. Observe that $\mathbf{G}_i \mathbf{H}_i^{\mathbf{T}} = \mathbf{0}_k$ for all $i = 1, 2, \ldots, \ell$.

For a general construction of Block-Matrix codes, the parity-check matrices of the constituent codes are redefined by multiplying with a suitable scalar multiple:

$$\mathcal{H}_i := \beta \mathbf{H}_i = \beta \begin{pmatrix} \mathbf{h}_i^{[0]} \\ \mathbf{h}_i^{[1]} \\ \vdots \\ \mathbf{h}_i^{[k-1]} \end{pmatrix} = \beta \begin{pmatrix} \alpha_{i1} & \alpha_{i2} & \cdots & \alpha_{in} \\ \alpha_{i1}^{[1]} & \alpha_{i2}^{[1]} & \cdots & \alpha_{in}^{[1]} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{i1}^{[k-1]} & \alpha_{i2}^{[k-1]} & \cdots & \alpha_{in}^{[k-1]} \end{pmatrix},$$

where $\beta \in \mathbb{F}^*_{\left(2^t\right)^{2k}}$. Clearly, $\mathbf{G}_i \mathcal{H}_i^{\mathbf{T}} = \mathbf{0}_k$ and $\mathcal{H}_i \mathcal{H}_i^{\mathbf{T}} = \beta \mathbf{H}_i \left(\beta \mathbf{H}_i\right)^{\mathbf{T}} = \beta^2 \left(\mathbf{H}_i \mathbf{H}_i^{\mathbf{T}}\right) = \beta^2 \mathbf{I}_k$ so that the resulting dual codes are trivial/non-trivial LCD MRD codes. Define the generator matrix $\mathbf{G}_{\mathrm{BM}}^{\beta}$ of Construction-I: General β construction as follows:

$$\mathbf{G}_{\mathrm{BM}}^{\beta} = \begin{pmatrix} \mathbf{G}_1 & \mathcal{H}_2 & \mathcal{H}_3 & \cdots & \mathcal{H}_\ell \\ \mathcal{H}_1 & \mathbf{G}_2 & \mathcal{H}_3 & \cdots & \mathcal{H}_\ell \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{H}_3 & \cdots & \mathbf{G}_\ell \end{pmatrix}.$$

Clearly, the $k\ell \times n\ell$ matrix $\mathbf{G}_{\mathrm{BM}}^{\beta}$ defines a $k\ell$-dimensional Block-Matrix code of length $n\ell$ with the minimum distance $d \leq n$, denoted by $\mathcal{C}_{\mathrm{BM}}^{\beta}\left(n^{\times \ell}, k^{\times \ell}, d\right)_{\left(2^t\right)^{2k}}$. As $\mathbf{G}_{\mathrm{BM}}^{\beta}$ is made up of generator matrices of LCD codes, it indeed defines an LCD code for certain conditions on β and α_i, $i = 1, 2, \ldots, \ell$. To prove that $\mathcal{C}_{\mathrm{BM}}^{\beta}$ is an LCD code, it is enough to show that $\mathbf{G}_{\mathrm{BM}}^{\beta} \mathbf{G}_{\mathrm{BM}}^{\beta \mathbf{T}}$ has a non-vanishing determinant. Using the underlying parity-check conditions $\mathbf{G}_i \mathbf{H}_i^{\mathbf{T}} = \mathbf{0}_k$ and $\mathbf{H}_i \mathbf{H}_i^{\mathbf{T}} = \mathbf{I}_k$ for $i = 1, 2, \ldots, \ell$, we have

$$\mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\,\mathbf{T}} = \begin{pmatrix} \mathbf{G}_1\mathbf{G}_1^{\mathrm{T}} + \beta^2 \sum_{j=2}^{\ell} \mathbf{H}_j\mathbf{H}_j^{\mathrm{T}} & \beta^2 \sum_{j=3}^{\ell} \mathbf{H}_j\mathbf{H}_j^{\mathrm{T}} & \cdots & \beta^2 \sum_{j=2}^{\ell-1} \mathbf{H}_j\mathbf{H}_j^{\mathrm{T}} \\ \beta^2 \sum_{j=3}^{\ell} \mathbf{H}_j\mathbf{H}_j^{\mathrm{T}} & \mathbf{G}_2\mathbf{G}_2^{\mathrm{T}} + \beta^2 \sum_{\substack{j=1 \\ j\neq 2}}^{\ell} \mathbf{H}_j\mathbf{H}_j^{\mathrm{T}} & \cdots & \beta^2 \sum_{\substack{j=1 \\ j\neq 2}}^{\ell-1} \mathbf{H}_j\mathbf{H}_j^{\mathrm{T}} \\ \vdots & \vdots & \ddots & \vdots \\ \beta^2 \sum_{j=2}^{\ell-1} \mathbf{H}_j\mathbf{H}_j^{\mathrm{T}} & \beta^2 \sum_{\substack{j=1 \\ j\neq 2}}^{\ell-1} \mathbf{H}_j\mathbf{H}_j^{\mathrm{T}} & \cdots & \mathbf{G}_\ell\mathbf{G}_\ell^{\mathrm{T}} + \beta^2 \sum_{j=1}^{\ell-1} \mathbf{H}_j\mathbf{H}_j^{\mathrm{T}} \end{pmatrix}$$

where the $(r,s)^{th}$ entry is a square matrix of order $k\times k$ given by

$$\mathcal{A}_{rs} := \begin{cases} \mathbf{G}_r\mathbf{G}_r^{\mathrm{T}} + \beta^2 \sum_{\substack{j=1 \\ j\neq r}}^{\ell} \mathbf{H}_j\mathbf{H}_j^{\mathrm{T}} & , \text{ for } r = s \\ \beta^2 \sum_{\substack{j=1 \\ j\neq r, j\neq s}}^{\ell} \mathbf{H}_j\mathbf{H}_j^{\mathrm{T}} & , \text{ for } r \neq s \end{cases}$$

$$= \begin{cases} \mathbf{G}_r\mathbf{G}_r^{\mathrm{T}} + (\ell-1)\beta^2\mathbf{I}_k & , \text{ for } r = s \\ (\ell-2)\beta^2\mathbf{I}_k & , \text{ for } r \neq s \end{cases}$$

with the diagonal-block $\mathcal{A}_{rr}$, in which the sum $\sum_{j}^{\ell} \mathbf{H}_j\mathbf{H}_j^{\mathrm{T}}$ is taken over all except $\mathbf{H}_r\mathbf{H}_r^{\mathrm{T}}\,(j = 1,2,\ldots,r-1,r+1,\ldots,\ell)$, whereas in off-diagonal blocks the sum $\sum_{j}^{\ell} \mathbf{H}_j\mathbf{H}_j^{\mathrm{T}}$ runs over all but except for $\mathbf{H}_r\mathbf{H}_r^{\mathrm{T}}$ and $\mathbf{H}_s\mathbf{H}_s^{\mathrm{T}}\,(j = 1,2,\ldots,r-1,r+1,\ldots,s-1,s+1,\ldots,\ell)$; in other words, due to $\mathbf{G}_i\mathbf{H}_i^{\mathrm{T}} = \mathbf{0}_k$, in r^{th} row-block, $\mathbf{H}_r\mathbf{H}_r^{\mathrm{T}}$ is not included, and in sth column-block $\mathbf{H}_s\mathbf{H}_s^{\mathrm{T}}$ is not included in the sum $\sum_{j}^{\ell} \mathbf{H}_j\mathbf{H}_j^{\mathrm{T}}$.

With regard to the number ℓ of constituent codes, the determinant of the block-matrix $\mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\mathbf{T}}$ is investigated under two cases in what follows.

3.3.2 Case (i): ℓ is Even

Clearly, $(\ell-1)\equiv 1(\mathrm{mod}2)$ and $(\ell-2)\equiv 0(\mathrm{mod}2)$ so that $\mathcal{A}_{rs} = \begin{cases} \mathbf{G}_r\mathbf{G}_r^{\mathrm{T}} + \beta^2\mathbf{I}_k & , \quad r = s \\ \mathbf{0}_k & , \quad r \neq s \end{cases}$.

Consequently, $\mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\,\mathrm{T}}$ is of block-diagonal form:

$$\mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\mathrm{T}} = \begin{pmatrix} \mathbf{G}_1\mathbf{G}_1^{\mathrm{T}} + \beta^2\mathbf{I}_k & 0_k & \cdots & 0_k \\ 0_k & \mathbf{G}_2\mathbf{G}_2^{\mathrm{T}} + \beta^2\mathbf{I}_k & \cdots & 0_k \\ \vdots & \vdots & \ddots & \vdots \\ 0_k & 0_k & \cdots & \mathbf{G}_\ell\mathbf{G}_\ell^{\mathrm{T}} + \beta^2\mathbf{I}_k \end{pmatrix}$$

Since $\mathbf{G}_i\mathbf{G}_i^{\mathrm{T}} = \mathrm{diag}\left(\alpha_i, \alpha_i^{[1]}, \ldots, \alpha_i^{[k-1]}\right)$ is diagonal, $\mathbf{G}_i\mathbf{G}_i^{\mathrm{T}} + \beta^2\mathbf{I}_k$ is again diagonal and consequently, $\mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\,\mathbf{T}}$ is a block-diagonal matrix, whose determinant is equal to the product of determinants of the diagonal blocks:

$$\left|\mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\,\mathrm{T}}\right| = \prod_{i=1}^{\ell}\left|\mathbf{G}_i\mathbf{G}_i^{\mathrm{T}} + \beta^2\mathbf{I}_k\right| = \prod_{i=1}^{\ell}\left(\prod_{j=0}^{k-1}\left(\alpha_i^{[j]} + \beta^2\right)\right),$$

where

$$\left|\mathbf{G}_i\mathbf{G}_i^{\mathrm{T}} + \beta^2\mathbf{I}_k\right| = \begin{vmatrix} \alpha_i + \beta^2 & 0 & \cdots & 0 \\ 0 & \alpha_i^{[1]} + \beta^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_i^{[k-1]} + \beta^2 \end{vmatrix}$$

Clearly, $\left|\mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\,\mathbf{T}}\right|$ is non-vanishing for all $\beta \in \mathbb{F}^{*}_{(2^t)^{2k}}$ such that β^2 is not a conjugate of α_i for $i = 1, 2, \ldots, \ell$.

3.3.3 Case (ii): ℓ is Odd

In this case, we have $(\ell - 1) \equiv 0 (\mathrm{mod}\, 2)$ and $(\ell - 2) \equiv 1 (\mathrm{mod}\, 2)$ with

$$\mathcal{A}_{rs} = \begin{cases} \mathbf{G}_r\mathbf{G}_r^{\mathrm{T}}, r = s \\ \beta^2\mathbf{I}_k, r \neq s \end{cases}$$

$$\text{Then } \mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\,\mathrm{T}} = \begin{pmatrix} \mathbf{G}_1\mathbf{G}_1^{\mathrm{T}} & \beta^2\mathbf{I}_k & \cdots & \beta^2\mathbf{I}_k \\ \beta^2\mathbf{I}_k & \mathbf{G}_2\mathbf{G}_2^{\mathrm{T}} & \cdots & \beta^2\mathbf{I}_k \\ \vdots & \vdots & \ddots & \vdots \\ \beta^2\mathbf{I}_k & \beta^2\mathbf{I}_k & \cdots & \mathbf{G}_\ell\mathbf{G}_\ell^{\mathrm{T}} \end{pmatrix}.$$

Note that the main-diagonal blocks are diagonal matrices and off-diagonal blocks are scalar matrices of the same order $k \times k$. Unlike the case of ℓ-even, for ℓ-odd, the matrix $\mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\,\mathbf{T}}$ is not block-diagonal. To find the conditions under which $\left|\mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\,\mathbf{T}}\right|$ is non-vanishing, we first derive a formula for obtaining its determinant. The previous non-diagonal matrix is systematically reduced to an upper-triangular block-matrix using block-wise row operations, the procedure of which is described in what follows:

Step I: Start by adding each block-row to the preceding block-row; that is, add ℓ^{th} row to $(\ell - 1)^{\mathrm{st}}$ row, add $(\ell - 1)^{\mathrm{st}}$ row to $(\ell - 2)^{\mathrm{nd}}$ row, and so on up to adding the second row to the first row. In this step, except for ℓ^{th} row and ℓ^{th} column, we obtain a $k(\ell - 1) \times k(\ell - 1)$ upper-triangular block-matrix.

Step II: Multiplying the ℓ^{th} block-row by the first block-matrix of the first row-block (i.e., by $\mathbf{G}_1\mathbf{G}_1^{\mathrm{T}} + \beta^2\mathbf{I}_k$) and then adding β^2 times the first block-row to the ℓ^{th} block-row, which results in the first block of the ℓ^{th} block-row as 0_k. Second, multiplying ℓ^{th} block-row by $\mathbf{G}_2\mathbf{G}_2^{\mathrm{T}} + \beta^2\mathbf{I}_k$ and then adding $\beta^2\sum_{i=1}^{2}\left(\mathbf{G}_i\mathbf{G}_i^{\mathrm{T}} + \beta^2\mathbf{I}_k\right)$ times the 2^{nd} block-row to it, the second block in ℓ^{th} block-row is reduced to 0_k. Proceeding in a similar manner to reduce each block of ℓ^{th} block-row to 0_k, till $(\ell-1)^{\text{st}}$ block is reduced to 0_k in ℓ^{th} block-row.

In the sequel, $\mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\mathbf{T}}$ is reduced to the following upper-triangular block-matrix:

$$\mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\mathbf{T}} \sim \begin{pmatrix} \mathbf{I}_k & \mathbf{G}_2\mathbf{G}_2^{\mathrm{T}} + \beta^2\mathbf{I}_k & 0_k & \cdots & 0_k \\ 0_k & \mathbf{I}_k & \mathbf{G}_3\mathbf{G}_3^{\mathrm{T}} + \beta^2\mathbf{I}_k & \cdots & 0_k \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0_k & 0_k & \cdots & \mathbf{I}_k & \mathbf{G}_\ell\mathbf{G}_\ell^{\mathrm{T}} + \beta^2\mathbf{I}_k \\ 0_k & 0_k & 0_k & \cdots & \mathbf{D}_\beta \end{pmatrix},$$

where $\mathbf{D}_\beta = \mathrm{diag}\left(d_0^\beta, d_1^\beta, \ldots, d_{k-1}^\beta\right)$

$$= \mathbf{G}_\ell\mathbf{G}_\ell^{\mathrm{T}}\left(\prod_{i=1}^{\ell-1}\left(\mathbf{G}_i\mathbf{G}_i^{\mathrm{T}} + \beta^2\mathbf{I}_k\right)\right) + \beta^2\left(\mathbf{G}_\ell\mathbf{G}_\ell^{\mathrm{T}} + \beta^2\mathbf{I}_k\right)$$

$$\sum_{j=0}^{\frac{\ell-1}{2}}\left(\sum_{i=2j+1}^{2(j+1)}\mathbf{G}_i\mathbf{G}_i^{\mathrm{T}} \prod_{\substack{i=1 \\ i\neq 2j+1,2(j+1)}}^{\ell-1}\left(\mathbf{G}_i\mathbf{G}_i^{\mathrm{T}} + \beta^2\mathbf{I}_k\right)\right)$$

$$= \mathrm{diag}\left(\alpha_\ell \prod_{i=1}^{\ell-1}\left(\alpha_i + \beta^2\right), \alpha_\ell^{[1]}\prod_{i=1}^{\ell-1}\left(\alpha_i^{[1]} + \beta^2\right), \ldots, \alpha_\ell^{[k-1]}\prod_{i=1}^{\ell-1}\left(\alpha_i^{[k-1]} + \beta^2\right)\right)$$

$$+ \quad \mathrm{diag}\left(\begin{array}{l} \beta^2\left(\alpha_\ell + \beta^2\right)\sum_{j=0}^{\frac{\ell-1}{2}}\left(\sum_{i=2j+1}^{2(j+1)}\alpha_i \prod_{\substack{i=1 \\ i\neq 2j+1,2(j+1)}}^{\ell-1}\left(\alpha_i + \beta^2\right)\right), \\ \beta^2\left(\alpha_\ell^{[1]} + \beta^2\right)\sum_{j=0}^{\frac{\ell-1}{2}}\left(\sum_{i=2j+1}^{2(j+1)}\alpha_i^{[1]} \prod_{\substack{i=1 \\ i\neq 2j+1,2(j+1)}}^{\ell-1}\left(\alpha_i^{[1]} + \beta^2\right)\right), \ldots, \\ \beta^2\left(\alpha_\ell^{[k-1]} + \beta^2\right)\sum_{j=0}^{\frac{\ell-1}{2}}\left(\sum_{i=2j+1}^{2(j+1)}\alpha_i^{[k-1]} \prod_{\substack{i=1 \\ i\neq 2j+1,2(j+1)}}^{\ell-1}\left(\alpha_i^{[k-1]} + \beta^2\right)\right) \end{array}\right)$$

is the $k\times k$ diagonal matrix, and hence $\left|\mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\,\mathrm{T}}\right|=\left|\mathbf{D}_{\beta}\right|$. The determinant $\left|\mathbf{D}_{\beta}\right|$ does not vanish as long as $d_j^{\beta}\neq 0$ for any j. Thus $\left|\mathbf{D}_{\beta}\right|\neq 0$ iff $\alpha_{\ell}^{[j]}\prod_{i=1}^{\ell-1}\left(\alpha_i^{[j]}+\beta^2\right)\neq\beta^2\left(\alpha_{\ell}^{[j]}+\beta^2\right)$ $\sum_{j=0}^{\frac{\ell-1}{2}}\left(\sum_{i=2j+1}^{2(j+1)}\alpha_i^{[j]}\prod_{\substack{i=1\\ i\neq 2j+1,2(j+1)}}^{\ell-1}\left(\alpha_i^{[j]}+\beta^2\right)\right)$ for $j=0,1,\ldots,k-1$. Further, the diagonal elements d_j^{β} of $\mathbf{D}_{\beta}=\mathrm{diag}\left(d_0^{\beta},d_1^{\beta},\ldots,d_{k-1}^{\beta}\right)$ are obtained as:

$$d_0^{\beta}=\alpha_{\ell}\prod_{i=1}^{\ell-1}\left(\alpha_i+\beta^2\right)+\beta^2\left(\alpha_{\ell}+\beta^2\right)\sum_{j=0}^{\frac{\ell-1}{2}}\left(\sum_{i=2j+1}^{2(j+1)}\alpha_i\prod_{\substack{i=1\\ i\neq 2j+1,2(j+1)}}^{\ell-1}\left(\alpha_i+\beta^2\right)\right)$$

$$d_1^{\beta}=\alpha_{\ell}^{[1]}\prod_{i=1}^{\ell-1}\left(\alpha_i^{[1]}+\beta^2\right)+\beta^2\left(\alpha_{\ell}^{[1]}+\beta^2\right)\times\sum_{j=0}^{\frac{\ell-1}{2}}\left(\sum_{i=2j+1}^{2(j+1)}\alpha_i^{[1]}\prod_{\substack{i=1\\ i\neq 2j+1,2(j+1)}}^{\ell-1}\left(\alpha_i^{[1]}+\beta^2\right)\right)$$

$$\vdots$$

$$d_{k-1}^{\beta}=\alpha_{\ell}^{[k-1]}\prod_{i=1}^{\ell-1}\left(\alpha_i^{[k-1]}+\beta^2\right)+\beta^2\left(\alpha_{\ell}^{[k-1]}+\beta^2\right)\times\sum_{j=0}^{\frac{\ell-1}{2}}\left(\sum_{i=2j+1}^{2(j+1)}\alpha_i^{[k-1]}\prod_{\substack{i=1\\ i\neq 2j+1,2(j+1)}}^{\ell-1}\left(\alpha_i^{[k-1]}+\beta^2\right)\right)$$

$$\therefore\left|\mathbf{G}_{BM}^{\beta}\mathbf{G}_{BM}^{\beta\mathbf{T}}\right|=\prod_{j=0}^{k-1}d_j^{\beta}=\prod_{j=0}^{k-1}\left(\begin{array}{l}\alpha_{\ell}^{[j]}\prod_{i=1}^{\ell-1}\left(\alpha_i^{[j]}+\beta^2\right)+\beta^2\left(\alpha_{\ell}^{[j]}+\beta^2\right)\\ \times\sum_{j=0}^{\frac{\ell-1}{2}}\left(\sum_{i=2j+1}^{2(j+1)}\alpha_i^{[j]}\prod_{\substack{i=1\\ i\neq 2j+1,2(j+1)}}^{\ell-1}\left(\alpha_i^{[j]}+\beta^2\right)\right)\end{array}\right).$$

Clearly, the choice of β such that β^2 is not a conjugate of α_i ensures that the determinant $\left|\mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\mathbf{T}}\right|$ does not vanish. Consequently, the Block-Matrix code $\mathcal{C}_{\mathrm{BM}}^{\beta}\left(n^{\times\ell},k^{\times\ell},d\right)_{\left(2^t\right)^{2k}}$ defines an LCD code, and the result of the construction is stated as a theorem.

Theorem 3.3.1 *Let* $n=2k$. *For* $\alpha_i\in\mathbb{F}^{*}_{\left(2^t\right)^{2k}}$, *the* Block-Matrix *code* $\mathcal{C}_{\mathrm{BM}}^{\beta}\left(n^{\times\ell},k^{\times\ell},d\right)_{\left(2^t\right)^{2k}}$ *is*

an LCD code iff $\begin{cases}\alpha_i^{[j]}\neq\beta^2 & \text{, when } \ell \text{ is even}\\ \alpha_{\ell}^{[j]}\prod_{i=1}^{\ell-1}\left(\alpha_i^{[j]}+\beta^2\right)\neq\beta^2\left(\alpha_{\ell}^{[j]}+\beta^2\right)\times & \\ \sum_{j=0}^{\frac{\ell-1}{2}}\left(\sum_{i=2j+1}^{2(j+1)}\alpha_i^{[j]}\prod_{i\neq 2j+1,2(j+1)}^{\ell-1}\left(\alpha_i^{[j]}+\beta^2\right)\right) & \text{, when } \ell \text{ is odd}\end{cases}$

where $\beta \in \mathbb{F}^{*}_{\left(2^t\right)^{2k}}$ *is such that* $\beta^2 \notin \left\{\alpha_i, \alpha_i^{[1]}, \ldots, \alpha_i^{[n-1]}\right\}$ *for* $i = 1, 2, \ldots, \ell$ *and* $j = 0, 1, 2, \ldots, k-1$.

Remark 3.3.1 *The proposed construction of Block-Matrix LCD codes* $\mathcal{C}_{\text{BM}}^{\beta}$ *employs* ℓ *constituent trivial (resp. non-trivial)* LCD MRD *codes and their associated* ℓ *dual non-trivial (resp. trivial)* LCD MRD *codes over* $\mathbb{F}_{\left(2^t\right)^{2k}}$ *that are constructed in accordance with Theorem 3.0.1. In particular, when* $\beta = 1$ *(resp.* $\beta \neq 1$*) and* $c_i \neq 1; \mathbf{G}_i\mathbf{G}_i^{\mathbf{T}} \neq \mathbf{I}_k$ *(resp.* $c_i = 1; \mathbf{G}_i\mathbf{G}_i^{\mathbf{T}} = \mathbf{I}_k$*), the generator matrix* $\mathbf{G}_i$ *describes a non-trivial (resp. trivial)* LCD MRD *code, whereas the corresponding parity-check matrix* $\mathcal{H}_i = \mathbf{H}_i$ *(resp.* $\mathcal{H}_i = \beta^2\mathbf{H}_i$*) describes a trivial (resp. non-trivial)* LCD MRD *code so that the constructed Block-Matrix code* $\mathcal{C}_{\text{BM}}^{\beta}$ *by* $\mathbf{G}_{\text{BM}}^{\beta}$ *is LCD as* $\left|\mathbf{G}_{\text{BM}}^{\beta}\mathbf{G}_{\text{BM}}^{\beta\,\mathbf{T}}\right| \neq 0$. *However, with* $\beta = 1$ *and* $c = 1$*, that is, using the trivial LCD MRD codes* $\left(\mathbf{G}_i\mathbf{G}_i^{\mathbf{T}} = \mathbf{H}_i\mathbf{H}_i^{\mathbf{T}} = \mathbf{I}_k\right)$*, the Block-Matrix code* $\mathcal{C}_{\text{BM}}^{\beta}$ *fails to be an LCD code, as* $\mathbf{G}_{\text{BM}}^{\beta}\mathbf{G}_{\text{BM}}^{\beta\,\mathbf{T}} = \mathbf{0}_{k\ell}$*! (for* $\beta = 1$ *and* $c_i = 1$ *because* $\mathbf{G}_i\mathbf{G}_i^{\mathbf{T}} + \sum \mathbf{H}_i\mathbf{H}_i^{\mathbf{T}} = (\mathbf{0})_k$ *and under both* ℓ*-even or for* ℓ*-odd cases).*

Remark 3.3.2 *While the purpose of this chapter is to investigate Block-Matrix LCD codes, it would be mentionable that another emerging class of self-dual codes* $\left(\mathcal{C} = \mathcal{C}^{\perp}\right)$ Glynn et al. (2007); Dougherty et al. (1999); Fuji-Hara and Miao (2000)*, of length* $k\ell$ *over* $\mathbb{F}_{\left(2^t\right)^{2k}}$ *can be obtained from the construction of Block-Matrix codes! An* LCD *code* $\mathcal{C}(n, k, d)$ *with generator matrix* $\mathbf{G}$ *requires* $\left|\mathbf{G}\mathbf{G}^{\mathbf{T}}\right| \neq 0$*, but the matrix* $\mathbf{G}\mathbf{G}^{\mathbf{T}}$ *need not be an all-zero matrix* $(\mathbf{0})_k$*. While for a code to be a self-dual code, the condition on* $\mathcal{C}$ *is that* $\mathbf{G}\mathbf{G}^{\mathbf{T}} = (\mathbf{0})_k$*. In Construction-I, when* ℓ *is even with* $c_i = 1$ *and* $\beta = 1$ *for* $1 \leq i \leq \ell$*, the constructed Block-Matrix code* $\mathcal{C}_{\text{BM}}^{\beta}$ *fails to be an* LCD *as* $\mathbf{G}_{\text{BM}}^{\beta}\mathbf{G}_{\text{BM}}^{\beta\,\mathbf{T}} = (\mathbf{0})_{k\ell}$*. On the other hand,* $\mathbf{G}_{\text{BM}}^{\beta}\mathbf{G}_{\text{BM}}^{\beta\,\mathbf{T}} = (\mathbf{0})_{k\ell}$ *implies that* $\mathcal{C}_{\text{BM}}^{\beta}$ *is a self-dual code!*

3.3.4 Construction-II: *q*-Cyclic β-Construction

The construction of Block-Matrix LCD codes $\mathcal{C}_{\text{BM}}$ of Kumar and Durai (2021) is in fact a particular case of the proposed general β-construction (Construction-I) of Block-Matrix LCD codes $\mathcal{C}_{\text{BM}}^{\beta}$; while the constituent codes of $\mathcal{C}_{\text{BM}}^{\beta}$ are obtained by ℓ generator matrices that are defined by ℓ *self-dual* bases, the constituent codes of $\mathcal{C}_{\text{BM}}$ are derived from ℓ generator matrices, which are generated by the q-cyclic shifts of a single *self-dual* basis. While the proposed Construction-I of Block-Matrix LCD codes require a maximum of ℓ *self-dual* bases of $\mathbb{F}_{\left(2^t\right)^{2k}}$, a special case involving a single *self-dual* basis of $\mathbb{F}_{\left(2^t\right)^{2k}}$ is presented in this subsection. This special case of Construction-I (*self-dual* β-construction) is in fact a general case of the *self-dual* Block-Matrix construction of Kumar and Durai (2021), in the sense that the construction proposed in Kumar and Durai (2021) discusses only the case when ℓ (number of constituent codes) is even. It is shown that it can be extended by employing a single *self-dual* basis to define constituent codes as well as their duals as non-trivial LCD MRD codes so that the special case also includes the construction proposed in Kumar and Durai (2021) and provides a general scenario. This general scenario is deduced as a particular case of Construction-I in what follows; the procedure is termed Construction-II.

Given a *self-dual* basis $\mathbf{h} = \{h_1, h_2, \ldots, h_n\}$ of $\mathbb{F}_{(2^t)^{2k}}$ and $\mathbf{g} = c\mathbf{h}^{[n-k]}$ for some $c \in \mathbb{F}^*_{(2^t)^{2k}}$, define the $i^{\text{th}} q = 2^t$-power of the generator matrix $\mathbf{G}$ as

$$\mathbf{G}^{(i)} = \begin{pmatrix} \mathbf{g}^{[i]} \\ \mathbf{g}^{[i+1]} \\ \vdots \\ \mathbf{g}^{[i+k-1]} \end{pmatrix}$$

for $i = 0, 1, \ldots, \ell - 1$. Let $\mathbf{H}^{(i)}$ be the $k \times n$ parity-check matrix satisfying $\mathbf{G}^{(i)}\mathbf{H}^{(i)\,\mathrm{T}} = 0_k$. Observe that $\mathbf{G}_1 = \mathbf{G}^{(0)}$ and $\mathbf{G}^{(0)}\mathbf{G}^{(0)\,\mathrm{T}} = \mathbf{G}_1\mathbf{G}_1^{\mathrm{T}} = \mathrm{diag}\left(\alpha, \alpha^{[1]}, \ldots, \alpha^{[k-1]}\right)$, where and hereafter $\alpha = c^2$, unless otherwise specified. For $c \neq 1, \mathbf{G}^{(i)}$ generates a *non-trivial* LCD MRD code $\mathcal{C}_i\left(n = 2k, k, d\right)$ (as in Kumar and Durai 2021) for each $i = 0, 1, \ldots, \ell - 1$. In general, $\mathbf{G}^{(i)}\mathbf{G}^{(i)\,\mathrm{T}} = \mathrm{diag}\left(\alpha^{[i]}, \alpha^{[i+1]}, \ldots, \alpha^{[i+k-1]}\right)$ and $\mathbf{H}^{(i)}\mathbf{H}^{(i)\,\mathrm{T}} = \mathbf{I}_k$. Define $\mathcal{H}^{(i)} := \beta\mathbf{H}^{(i)}$ for some $\beta \in \mathbb{F}^*_{(2^t)^{2k}}$. In the generator matrix $\mathbf{G}^{\beta}_{\mathrm{BM}}$ of Construction-I, set $\mathbf{G}_i = \mathbf{G}^{(i-1)}$ and $\mathcal{H}_i = \mathcal{H}^{(i-1)}$ so that the generator matrix $\mathbf{G}^{q\beta}_{\mathrm{BM}}$ of Construction-II: q-cyclic β-construction is defined as follows:

$$\mathbf{G}^{q\beta}_{\mathrm{BM}} := \begin{pmatrix} \mathbf{G}^{(0)} & \mathcal{H}^{(1)} & \mathcal{H}^{(2)} & \ldots & \mathcal{H}^{(\ell-1)} \\ \mathcal{H}^{(0)} & \mathbf{G}^{(1)} & \mathcal{H}^{(2)} & \ldots & \mathcal{H}^{(\ell-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{H}^{(0)} & \mathcal{H}^{(1)} & \mathcal{H}^{(2)} & \ldots & \mathbf{G}^{(\ell-1)} \end{pmatrix}$$

is composed of ℓ *non-trivial* LCD MRD codes with the corresponding *non-trivial* LCD MRD dual codes, defined by the q-powers of the *self-dual* basis $\mathbf{h} = \{h_1, h_2, \ldots, h_n\}$. Let $\mathcal{C}^{q\beta}_{\mathrm{BM}}\left(n^{\times\ell}, k^{\times\ell}, d\right)_{(2^t)^{2k}}$ denote the resulting Block-Matrix code defined by $\mathbf{G}^{q\beta}_{\mathrm{BM}}$. As $\mathbf{G}_i = \mathbf{G}^{(i-1)}$ and $\mathcal{H}_i = \mathcal{H}^{(i-1)}$, the code $\mathcal{C}^{q\beta}_{\mathrm{BM}}$ generated by $\mathbf{G}^{q\beta}_{\mathrm{BM}}$ can be shown to be an LCD code analogously. Consequently, the necessary and sufficient condition for the Block-Matrix code $\mathcal{C}^{q\beta}_{\mathrm{BM}}\left(n^{\times\ell}, k^{\times\ell}, d\right)_{(2^t)^{2k}}$ to have complementary dual is stated in the following theorem.

Theorem 3.3.2 *Let $n = 2k$. For $\alpha \in \mathbb{F}^*_{(2^t)^{2k}}$, the* Block-Matrix *code $\mathcal{C}^{q\beta}_{\mathrm{BM}}\left(n^{\times\ell}, k^{\times\ell}, d\right)_{(2^t)^{2k}}$ is an*

$$\text{LCD } \textit{code iff } \begin{cases} \alpha \neq \beta^2 & , \ell \text{ is odd} \\ \alpha^{[\ell-1+j]} \prod_{i=0}^{\ell-2} \left(\alpha^{[i+j+1]} + \beta^2\right) \neq \beta^2 \left(\alpha^{[\ell-1+j]} + \beta^2\right) \times & , \\ \quad \sum_{j=0}^{\frac{\ell-2}{2}} \left(\sum_{i=2j}^{2j+1} \alpha^{[i+j+1]} \prod_{\substack{i=0 \\ i \neq 2j, 2j+1}}^{\ell-2} \left(\alpha^{[i+j+1]} + \beta^2\right) \right) & , \ell \text{ is even} \end{cases}$$

where $\beta \in \mathbb{F}^{*}_{\left(2^t\right)^{2k}}$ *is such that* $\beta^2 \notin \left\{\alpha, \alpha^{[1]}, \ldots, \alpha^{[n-1]}\right\}$ *for* $j = 0,1,2,\ldots,k-1$.

Notice that in Construction-II, for ℓ odd the number of constituent codes is even and for ℓ even the number of constituent codes is odd. The Block-Matrix LCD codes construction proposed in Kumar and Durai (2021) can be obtained as a particular case of (Construction-II) with $\beta = 1$!

Remark 3.3.3 *It must be emphasized that the choice of the field* $\mathbb{F}_{\left(2^t\right)^n}$ *with* $n = 2k$ *is necessary for the addition of* $k \times k$ *matrix* $\mathbf{G}_i\mathbf{G}_i^{\mathrm{T}}$ *and* $(n-k)\times k$ *matrix* $\mathbf{H}_j\mathbf{H}_j^{\mathrm{T}}$ *to be conformable while computing the* $k\ell \times k\ell$ *matrix-product* $\mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\mathbf{T}}$. *Although the characteristic of the underlying field* $\mathbb{F}_{\left(p^t\right)^{2k}}$ *can be any prime* p, *in the present work, we discuss only the case of* $p = 2$ *to obtain a specific form of* $\mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\mathbf{T}}$.

3.4 THE PARITY-CHECK MATRIX OF BLOCK-MATRIX CODES

The $k\ell \times n\ell$ parity-check matrix $\mathbf{H}_{\mathrm{BM}}$ for a Block-Matrix code $\mathcal{C}_{\mathrm{BM}}\left(n^{\times\ell}, k^{\times\ell}, d\right)_{\left(2^t\right)^{2k}}$ can be obtained from $\mathbf{G}_{\mathrm{BM}}$ by solving a system of $k\ell$ equations. One way to obtain $\mathbf{H}_{\mathrm{BM}}$ is to consider it to be the matrix of the form

$$\begin{pmatrix} \mathbf{G}_1 & \mathrm{X}'_{12} & \cdots & \mathrm{X}'_{1\ell} \\ \mathrm{X}'_{21} & \mathbf{G}_2 & \cdots & \mathrm{X}'_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{X}'_{\ell 1} & \mathrm{X}'_{\ell 2} & \cdots & \mathbf{G}_\ell \end{pmatrix}$$

where X'_{ij} is a $k \times n$ matrix over $\mathbb{F}_{\left(2^t\right)^{2k}}$ for $i = 1,2,\ldots,\ell$. To obtain $\mathbf{H}_{\mathrm{BM}}$, each row-block (out of ℓ) is considered exclusively to solve the system of k linear equations over $\mathbb{F}_{\left(2^t\right)^{2k}}$. Let $\mathbf{H}_{\mathrm{BM}}^{\mathrm{T}}(i)$ denote the transpose of the i^{th} row-block $\left(\mathrm{X}'_{i1}, \mathrm{X}'_{i2}, \ldots, \mathrm{X}'_{i(j-1)}, \mathbf{G}_j, \mathrm{X}'_{i(j+1)}, \ldots, \mathrm{X}'_{i\ell}\right)$ of $\mathbf{H}_{\mathrm{BM}}$. Fix the i^{th} constituent generator matrix $\mathbf{G}_i$ at i^{th} position in each row-block and then solve the system of equations $\mathbf{G}_{\mathrm{BM}}\mathbf{H}_{\mathrm{BM}}^{\mathrm{T}}(i) = \mathbf{0}_{k\ell\times k}$ The systematically obtained parity-check matrix $\mathbf{H}_{\mathrm{BM}}$ is also of full rank satisfying $\mathbf{G}_{\mathrm{BM}}\mathbf{H}_{\mathrm{BM}}^{\mathrm{T}} = \mathbf{0}_{k\ell}$

Example 3.4.1 *For* $\ell = k = 2$ *with* $t = \beta = 1$, *consider the Block-Matrix code* $\mathcal{C}_{\mathrm{BM}}^{q\beta}\left(4^{\times 2}, 2^{\times 2}, 3\right)_{2^4}$ *defined by the following generator and parity-check matrices:*

$$\mathbf{G}_1 = \begin{pmatrix} \alpha^8 & \alpha^{10} & \alpha^5 & \alpha^{13} \\ \alpha & \alpha^5 & \alpha^{10} & \alpha^{11} \end{pmatrix}; \qquad \mathbf{H}_1 = \begin{pmatrix} \alpha^6 & \alpha^{14} & \alpha^9 & \alpha^{11} \\ \alpha^{12} & \alpha^{13} & \alpha^3 & \alpha^7 \end{pmatrix}$$

$$\text{and } \mathbf{G}_2 = \mathbf{G}_1^{[1]}; \qquad \mathbf{H}_2 = \mathbf{H}_1^{[1]}.$$

Clearly, $\mathbf{G}_{\mathrm{BM}} = \begin{pmatrix} \mathbf{G}_1 & \mathbf{H}_2 \\ \mathbf{H}_1 & \mathbf{G}_2 \end{pmatrix}$ and $\mathbf{H}_{\mathrm{BM}} = \begin{pmatrix} \mathbf{G}_1 & \mathrm{X}'_{12} \\ \mathrm{X}'_{21} & \mathbf{G}_2 \end{pmatrix}$,

where

$$\mathrm{x}'_{12} = \begin{pmatrix} \alpha^{10} & \alpha^{11} & \alpha & \alpha^{5} \\ \alpha^{5} & \alpha^{7} & \alpha^{2} & \alpha^{10} \end{pmatrix}; \mathrm{x}'_{21} = \begin{pmatrix} \alpha^{2} & \alpha^{10} & \alpha^{5} & \alpha^{7} \\ \alpha^{4} & \alpha^{5} & \alpha^{10} & \alpha^{14} \end{pmatrix}$$

and $\mathbf{G}_{\mathrm{BM}}\mathbf{H}_{\mathrm{BM}}^{\mathrm{T}} = \mathbf{0}_4$. Also, $\mathrm{Rk}\left(\mathbf{G}_{\mathrm{BM}}\right) = 4 = \mathrm{Rk}\left(\mathbf{H}_{\mathrm{BM}}\right)$.

3.5 CONSTRUCTION OF BLOCK-MATRIX CODES OVER $\mathbb{F}_{\left(2^t\right)^{2k+1}}$

All the proposed constructions of the Block-Matrix LCD codes are defined over the field $\mathbb{F}_{\left(2^t\right)^{2k}}$ only. This section attempts to construct a class of Block-Matrix LCD codes over $\mathbb{F}_{\left(2^t\right)^{2k+1}}$. The required constituent codes of dimension k over $\mathbb{F}_{\left(2^t\right)^{2k+1}}$ are obtained in a similar manner as Block-Matrix LCD codes defined over $\mathbb{F}_{\left(2^t\right)^{2k}}$; however, the parity-check matrices corresponding to the constituent codes are modified.

To construct a Block-Matrix LCD code $\mathcal{C}_{\mathrm{BM}}^{\beta}\left(n^{\times\ell}, k^{\times\ell}, d\right)$ over $\mathbb{F}_{\left(2^t\right)^n}$ for $n = 2k+1$ generated by $\mathbf{G}_{\mathrm{BM}}^{\beta}$ (of order $k\ell \times (2k+1)\ell$), ℓ *self-dual* bases $\mathbf{h}_1, \mathbf{h}_1, \ldots, \mathbf{h}_\ell$ of $\mathbb{F}_{\left(2^t\right)^{2k+1}}$. Then, as per Construction-I, the generator matrices $\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_\ell$ will be of order $k \times n$, but the associated parity-check matrices $\mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_\ell$ will be of order $(k+1) \times n$ so that this results in a situation where the generator and parity-check matrices are non-conformable for matrix multiplication while attempting to obtain $\mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\mathbf{T}}$. To make the matrix multiplication conformable, consider the k-dimensional constituent codes $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\ell$ over $\mathbb{F}_{\left(2^t\right)^{2k+1}}$ defined, respectively, by the generator matrices $\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_\ell$, and let $\mathbf{H}'_i$ denote the $k \times n$ matrix obtained, respectively, by *truncating* the $(k+1) \times n$ matrix $\mathbf{H}_i$ first k rows $\mathbf{h}_i^{[0]}, \mathbf{h}_i^{[1]}, \ldots, \mathbf{h}_i^{[k-1]}$ so that

$$\mathbf{G}_i = \begin{pmatrix} \mathbf{g}_i^{[0]} \\ \mathbf{g}_i^{[1]} \\ \vdots \\ \mathbf{g}_i^{[k-1]} \end{pmatrix} \text{ and } \mathbf{H}'_i = \begin{pmatrix} \mathbf{h}_i^{[0]} \\ \mathbf{h}_i^{[1]} \\ \vdots \\ \mathbf{h}_i^{[n-k-2]} \end{pmatrix}$$

where $\mathbf{g}_i^{[0]} = c_i \mathbf{h}_i^{[n-k]}$ for some $c_i \in \mathbb{F}^*_{\left(2^t\right)^{2k+1}} \setminus \{1\}$ for $i = 1, 2, \ldots, \ell$. Clearly, $\mathbf{G}_i\mathbf{G}_i^{\mathrm{T}} = \mathrm{diag}\left(\left(c_i{}^2\right), \left(c_i{}^2\right)^{[1]}, \ldots, \left(c_i{}^2\right)^{[k-1]}\right)$ and $\mathbf{H}'_i\mathbf{H}_i'^{\mathrm{T}} = \mathbf{I}_k$ for each i. Then the ℓ LCD MRD codes defined by the generator matrices constitute a Block-Matrix LCD code $\mathcal{C}_{\mathrm{BM}}^{\beta}\left(n^{\times\ell}, k^{\times\ell}, d\right)$ over $\mathbb{F}_{\left(2^t\right)^{2k+1}}$. Hence, the construction of the Block-Matrix LCD codes can be defined over $\mathbb{F}_{\left(2^t\right)^n}$ for any n ($2k$ or $2k+1$). The following example illustrates the *truncation* of the parity-check matrix for an LCD MRD code.

Example 3.5.1 *For* $t = 1, k = 2$ *and* $n = 2k+1 = 5$, *consider* $\mathbb{F}_{2^5}$ *with the self-dual basis* $\mathbf{h} = \left(\alpha^{11}, \alpha^{22}, \alpha^{13}, \alpha^{26}, \alpha^{21}\right)$, *where* $1 + \alpha^3 + \alpha^5 = 0$. *For* $k = 2$, *the* 2×5 *generator matrix*

$\mathbf{G}$ *of the constituent LCD MRD code* $\mathcal{C}(5,2,4)$ *and the corresponding* 3×5 *parity-check matrix* $\mathbf{H}$ *are given by*

$$\mathbf{G}=\begin{pmatrix}\alpha^{26} & \alpha^{21} & \alpha^{11} & \alpha^{22} & \alpha^{13}\\ \alpha^{21} & \alpha^{11} & \alpha^{22} & \alpha^{13} & \alpha^{26}\end{pmatrix};\ \mathbf{H}=\begin{pmatrix}\alpha^{11} & \alpha^{22} & \alpha^{13} & \alpha^{26} & \alpha^{21}\\ \alpha^{22} & \alpha^{13} & \alpha^{26} & \alpha^{21} & \alpha^{11}\\ \alpha^{13} & \alpha^{26} & \alpha^{21} & \alpha^{11} & \alpha^{22}\end{pmatrix}.$$

Clearly, $\mathbf{G}\mathbf{G}^{\mathrm{T}}=\mathbf{I}_2, \mathbf{H}\mathbf{H}^{\mathrm{T}}=\mathbf{I}_3$ *and* $\mathbf{G}\mathbf{H}^{\mathrm{T}}=\mathbf{0}_{2\times 3}$ *Then the* 2×5 *truncated parity-check matrix* $\mathbf{H}'$ *is obtained by omitting the last row of* $\mathbf{H}$ *as*

$$\mathbf{H}'=\begin{pmatrix}\alpha^{11} & \alpha^{22} & \alpha^{13} & \alpha^{26} & \alpha^{21}\\ \alpha^{22} & \alpha^{13} & \alpha^{26} & \alpha^{21} & \alpha^{11}\end{pmatrix}$$

so that $\mathbf{H}'\mathbf{H}'^{\mathrm{T}}=\mathbf{I}_2$ *and* $\mathbf{G}\mathbf{H}'^{\mathrm{T}}=\mathbf{0}_2$. *Notice that* $\mathcal{C}\oplus\mathcal{C}^{\perp}=\mathbb{F}_{2^5}^5$ *and* $\mathcal{C}\oplus\mathcal{C}'^{\perp}=\mathbf{F}_{2^5}^4\subset\mathbb{F}_{2^5}^5$, *where* $\mathcal{C}'$ *denotes the dual of* $\mathcal{C}$ *generated by* $\mathbf{H}'$ *and* $\mathbf{F}_{2^5}^4$ *denotes the four-dimensional subspace of the five-dimensional vector space* $\mathbb{F}_{2^5}^5$ *spanned by* $\mathbf{h}^{[0]},\mathbf{h}^{[1]},\mathbf{h}^{[3]},\mathbf{h}^{[4]}$.

Remark 3.3.4 *While the class of Self-dual codes can be obtained from the construction of Block-Matrix codes over* $\mathbb{F}_{\left(2^t\right)^{2k}}$ *as mentioned in Remark 3.3.3, in a similar manner, self-orthogonal codes* ($\mathcal{C}\subseteq\mathcal{C}^{\perp}$) *can be obtained from Block-Matrix codes constructed over* $\mathbb{F}_{\left(2^t\right)^{2k+1}}$; *when* ℓ *is even with* $c_i=1$ *and* $\beta=1$ *for* $1\le i\le\ell$, *the constructed Block-Matrix code* $\mathcal{C}_{\mathrm{BM}}^{\beta}$ *fails to be LCD as* $\mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\mathbf{T}}=(\mathbf{0})_{k\ell}$. *On the other hand,* $\mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\,\mathbf{T}}=(\mathbf{0})_{k\ell}$ *implies that* $\mathcal{C}_{\mathrm{BM}}^{\beta}$ *is a self-orthogonal code!*

3.6 A SIMPLE APPLICATION

The constructed class of Block-Matrix LCD codes can be used for encoding and decoding user messages of length $k\ell$ over a $\left(2^t\right)^{2k}$-ary channel. Assume that the channel is noiseless. Consider the general β-construction of Block-Matrix LCD code $\mathcal{C}_{\mathrm{BM}}^{\beta}$ composed of ℓk-dimensional *non-trivial* LCD MRD codes $\mathcal{C}_1(n=2k,k,d),\mathcal{C}_2(n=2k,k,d),\ldots,\mathcal{C}_\ell(n=2k,k,d)$ with the respective generator matrices $\mathbf{G}_1,\mathbf{G}_2,\ldots,\mathbf{G}_\ell$ and parity-check matrices $\mathcal{H}_1,\mathcal{H}_2,\ldots,\mathcal{H}_\ell$. Then we have $\mathbf{G}_i\mathbf{G}_i^{\mathrm{T}}=\mathbf{diag}\left(\alpha_i{}^{[0]},\alpha_i{}^{[1]},\ldots,\alpha_i{}^{[k-1]}\right)$ and $\mathcal{H}_i\mathcal{H}_i^{\mathrm{T}}=\beta^2\mathbf{I}_k$ such that β^2 is not a conjugate of α_i. The encoding and decoding procedures are given in what follows.

3.6.1 Encoding

Consider the message-block $\mathbf{m}_{\mathrm{BM}}=(\mathbf{m}_1,\mathbf{m}_2,\ldots,\mathbf{m}_\ell)$ of length $k\ell$ that is to be transmitted over a $\left(2^t\right)^{2k}$-ary noiseless channel, where the i^{th} segment $\mathbf{m}_i$ is $(m_{i1},m_{i2},\ldots,m_{ik})\in\mathbb{F}_{\left(2^t\right)^{2k}}^k$.

The vector-block $\mathbf{m}_{\mathrm{BM}}$ is encoded through $\mathbf{G}_{\mathrm{BM}}^{\beta}$, and the $n\ell$-length codeword $\mathbf{c}_{\mathrm{BM}} = \mathbf{m}_{\mathrm{BM}}\mathbf{G}_{\mathrm{BM}}^{\beta} = (\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_\ell) \in \mathcal{C}_{\mathrm{BM}}$ is obtained with

$$\mathbf{c}_i = \mathbf{m}_i\mathbf{G}_i + \sum_{i \neq j} \mathbf{m}_j \mathcal{H}_i = \tilde{\mathbf{c}}_i + \sum_{i \neq j} \mathbf{m}_j \mathcal{H}_i,$$

where $\tilde{\mathbf{c}}_i \in \mathcal{C}_i\,(n = 2k, k, d)$ and $\mathbf{c}_i = (c_{i1}, c_{i2}, \ldots, c_{in}) \in \mathbb{F}_{(2^t)^{2k}}^{n}$ is the i^{th} codeword corresponding to i^{th} column-block of $\mathbf{G}_{\mathrm{BM}}^{\beta}$.

3.6.2 Decoding

Let the received codeword be $\mathbf{c}_{\mathrm{BM}} = (\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_\ell)$, as the channel is noiseless. The transmitted message $\mathbf{m}_{\mathrm{BM}} = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_\ell)$ is then recovered by using the *orthogonal projector* $\Pi_{\mathcal{C}_i} = \mathbf{G}_i^{\mathrm{T}}\left(\mathbf{G}_i\mathbf{G}_i^{\mathrm{T}}\right)^{-1}\mathbf{G}_i$ of the constituent code $\mathcal{C}_i$ to each column-codeword $\mathbf{c}_i$ as

$$\mathbf{c}_i\Pi_{\mathcal{C}_i} = \left(\tilde{\mathbf{c}}_i + \sum_{i \neq j} m_j \mathcal{H}_i\right)\Pi_{\mathcal{C}_i} = \tilde{\mathbf{c}}_i\,\Pi_{\mathcal{C}_i} + \left(\sum_{i \neq j} \mathbf{m}_j \mathcal{H}_i\right)\Pi_{\mathcal{C}_i} = \tilde{\mathbf{c}}_i, \text{ for} 1 \leq i \leq \ell$$

Since $\sum_{i \neq j} \mathbf{m}_j \mathcal{H}_i \in \mathcal{C}_i^{\perp}, \mathbf{m}_i$ is recovered from $\tilde{\mathbf{c}}_i$ through the i^{th}-constituent code $\mathcal{C}_i$ conventionally. A collective *orthogonal projector* for decoding the noiseless vector $\mathbf{c}_{\mathrm{BM}} \in \mathcal{C}_{\mathrm{BM}}^{\beta}$ can also be defined as

$$\Pi_{\mathcal{C}_{\mathrm{BM}}^{\beta}} = \begin{pmatrix} \Pi_{\mathbf{G}_1} & 0_k & \cdots & 0_k \\ 0_k & \Pi_{\mathbf{G}_2} & \cdots & 0_k \\ \vdots & \vdots & \ddots & \vdots \\ 0_k & 0_k & \cdots & \Pi_{\mathbf{G}_\ell} \end{pmatrix}$$

so that $\mathbf{c}_{\mathrm{BM}}\Pi_{\mathcal{C}_{\mathrm{BM}}^{\beta}} = (\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_\ell)\Pi_{\mathcal{C}_{\mathrm{BM}}^{\beta}} = \left(\tilde{\mathbf{c}}_1, \tilde{\mathbf{c}}_2, \ldots, \tilde{\mathbf{c}}_\ell\right)$, and $\mathbf{m}_i$ is obtained from $\mathbf{m}_i\mathbf{G}_i = \tilde{\mathbf{c}}_i$ for each $i = 1, 2, \ldots, \ell$.

3.7 EXAMPLES OF BLOCK-MATRIX LCD CODES

This section presents several examples to demonstrate the proposed class of Block-Matrix LCD codes over $\mathbb{F}_{(2^t)^{2k}}$. For distinction purpose - with respect to the *trivial* and *non-trivial* constituent codes, the generator matrix of the Block-Matrix LCD code $\mathcal{C}_{\mathrm{BM}}^{\beta}$ is described under three cases of $\beta = \beta_0, \beta_1, \beta_2$:

$$\mathbf{G}_{\mathrm{BM}}^{\beta} = \begin{pmatrix} \mathbf{G}_1 & \beta\mathbf{H}_2 & \beta\mathbf{H}_3 & \cdots & \beta\mathbf{H}_\ell \\ \beta\mathbf{H}_1 & \mathbf{G}_2 & \beta\mathbf{H}_3 & \cdots & \beta\mathbf{H}_\ell \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta\mathbf{H}_1 & \beta\mathbf{H}_2 & \beta\mathbf{H}_3 & \cdots & \mathbf{G}_\ell \end{pmatrix} = \begin{cases} \mathbf{G}_{\mathrm{BM}}^{\beta_0} & , \quad c_i = 1 \text{and} \beta_0 = 1 \\ \mathbf{G}_{\mathrm{BM}}^{\beta_1} & , \quad c_i \neq 1 \text{and} \beta_1 = 1 \\ \mathbf{G}_{\mathrm{BM}}^{\beta_2} & , \quad c_i \neq 1 \text{and} \beta_2 \neq 1 \end{cases}$$

where $\mathbf{g}_i^{[0]} = c_i \mathbf{h}_i^{[n-k]}$ for $c_i \in \mathbb{F}^*_{\left(2^t\right)^{2k}}$ and $i = 1,2,\ldots,\ell$. The three cases can be described in terms of the combinations of constituent LCD MRD codes and their dual LCD MRD codes as being *trivial* and *non-trivial*:

- $\mathbf{G}_{\mathrm{BM}}^{\beta_0}$: both constituent codes and their corresponding duals are *trivial*; $\mathcal{C}_{\mathrm{BM}}^{\beta_0}$ defined by $\mathbf{G}_{\mathrm{BM}}^{\beta_0}$ is LCD.
- $\mathbf{G}_{\mathrm{BM}}^{\beta_1}$: the constituent codes are *non-trivial* codes, but duals are *trivial* codes; $\mathcal{C}_{\mathrm{BM}}^{\beta_1}$ defined by $\mathbf{G}_{\mathrm{BM}}^{\beta_1}$ is not LCD.
- $\mathbf{G}_{\mathrm{BM}}^{\beta_2}$: both constituent codes and their corresponding duals are *non-trivial* codes; $\mathcal{C}_{\mathrm{BM}}^{\beta_2}$ defined by $\mathbf{G}_{\mathrm{BM}}^{\beta_2}$ is LCD.
- Observe that the Block-Matrix code $\mathcal{C}_{\mathrm{BM}}^{\beta}$ defined by $\mathbf{G}_{\mathrm{BM}}^{\beta}$ is LCD (Construction-I and II) for $\beta = \beta_1$ and β_2 and not LCD for $\beta = \beta_0$.

3.7.1 Examples of General β-Construction: Construction-I

Example 3.7.1 *Examples over* $\mathbb{F}_{2^4}$: Let $t = 1, k = 2$, and $n = 2k = 4$. With $\ell = 2,3,4$, consider the *self-dual* bases: $\mathbf{h}_1 = \left(\alpha^4, \alpha, \alpha^6, \alpha^9\right), \mathbf{h}_2 = \left(\alpha^2, \alpha^3, \alpha^8, \alpha^{12}\right), \mathbf{h}_3 = \left(\alpha^8, \alpha^2, \alpha^{12}, \alpha^3\right), \mathbf{h}_4 = \left(\alpha^8, \alpha^{12}, \alpha^2, \alpha^3\right)$ from $\mathbb{F}_{2^4}$, where $1 + \alpha^3 + \alpha^4 = 0$. Let $\mathbf{g}_i^{[0]} = c_i \mathbf{h}_i^{[2]}$ for $i = 1,2,3,4$ with $c_1 = \alpha, c_2 = \alpha^3, c_3 = \alpha^{11}, c_4 = \alpha^7$. Further, consider the required 2×4 generator and the parity-check matrices of ℓ LCD MRD constituent codes:

$$
\begin{aligned}
\mathbf{G}_1 &= \left\langle \alpha^2 \quad \alpha^5 \quad \alpha^{10} \quad \alpha^7 \right\rangle_2; & \mathbf{H}_1 &= \left\langle \alpha^4 \quad \alpha \quad \alpha^6 \quad \alpha^9 \right\rangle_2 \\
\mathbf{G}_2 &= \left\langle \alpha^{11} \quad 1 \quad \alpha^5 \quad \alpha^6 \right\rangle_2; & \mathbf{H}_2 &= \left\langle \alpha^2 \quad \alpha^3 \quad \alpha^8 \quad \alpha^{12} \right\rangle_2 \\
\mathbf{G}_3 &= \left\langle \alpha^{13} \quad \alpha^4 \quad \alpha^{14} \quad \alpha^8 \right\rangle_2; & \mathbf{H}_3 &= \left\langle \alpha^8 \quad \alpha^2 \quad \alpha^{12} \quad \alpha^3 \right\rangle_2 \\
\mathbf{G}_4 &= \left\langle \alpha^9 \quad \alpha^{10} \quad 1 \quad \alpha^4 \right\rangle_2; & \mathbf{H}_4 &= \left\langle \alpha^8 \quad \alpha^{12} \quad \alpha^2 \quad \alpha^3 \right\rangle_2.
\end{aligned}
$$

The determinants $\left|\mathbf{G}_{\mathrm{BM}}^{\beta} \mathbf{G}_{\mathrm{BM}}^{\beta\,\mathrm{T}}\right|$ for $\beta_1 = 1$ and $\beta_2 = \alpha^5$ are tabulated in Table 3.1 for several values of ℓ. Thus, from the determinant values, it is clear that

Table 3.1 Determinant Values of $\mathrm{G}_{\mathrm{BM}}^{\beta_1}$ $\mathrm{G}_{\mathrm{BM}}^{\beta_1\mathrm{T}}$ and $\mathrm{G}_{\mathrm{BM}}^{\beta_2}$ $\mathrm{G}_{\mathrm{BM}}^{\beta_2\mathrm{T}}$

ℓ	$\left\lvert \boldsymbol{G}_{BM}^{\beta_1} \boldsymbol{G}_{BM}^{\beta_1 T} \right\rvert$	$\left\lvert \boldsymbol{G}_{BM}^{\beta_2} \boldsymbol{G}_{BM}^{\beta_2 T} \right\rvert$
2	α^6	α^3
3	α^3	α^7
4	α^3	α^9

$\mathcal{C}_{\mathrm{BM}}^{\beta_1}$ and $\mathcal{C}_{\mathrm{BM}}^{\beta_2}$ are LCD codes. However, for $\beta = \beta_0$, the constituent codes and their duals are trivial LCD MRD codes (i.e., $\beta_0 = 1$), we have $\left|\mathbf{G}_{\mathrm{BM}}^{\beta_0}\mathbf{G}_{\mathrm{BM}}^{\beta_0\mathrm{T}}\right| = 0$ for $2 \le \ell \le 4$, and hence $\mathcal{C}_{\mathrm{BM}}^{\beta_0}$ is not an LCD code. Also, for the even values of $\ell = 2,4$, we have $\mathbf{G}_{\mathrm{BM}}^{\beta_0}\mathbf{G}_{\mathrm{BM}}^{\beta_0\mathrm{T}} = \mathbf{0}_{2\ell}$.

Example 3.7.2 *Examples over* $\mathbb{F}_{2^6}$*: Let* $t = 1, k = 3$ *and* $n = 2k = 6$. *With* $\ell = 2,3,4,5,6$, *consider the following self-dual bases from* $\mathbb{F}_{2^6}$: $\mathbf{h}_1 = \left(\alpha^2, \alpha^5, \alpha^{13}, \alpha^{23}, \alpha^{26}, \alpha^{34}\right), \mathbf{h}_2 = \left(\alpha, \alpha^6, \alpha^8, \alpha^{29}, \alpha^{43}, \alpha^{48}\right), \mathbf{h}_3 = \left(\alpha^3, \alpha^4, \alpha^{24}, \alpha^{32}, \alpha^{46}, \alpha^{53}\right), \mathbf{h}_4 = \left(\alpha^{10}, \alpha, \alpha^8, \alpha^{17}, \alpha^{46}, \alpha^{53}\right), \mathbf{h}_5 = \left(\alpha^{16}, \alpha^2, \alpha^{12}, \alpha^{23}, \alpha^{33}, \alpha^{58}\right)$, *and* $\mathbf{h}_6 = \left(\alpha^{19}, \alpha^3, \alpha^6, \alpha^8, \alpha^{41}, \alpha^{53}\right)$, *where* $1 + \alpha^5 + \alpha^6 = 0$. *Let* $\mathbf{g}_i^{[0]} = c_i \mathbf{h}_i^{[3]}$ *for* $i = 1,2,3,4,5,6$ *with* $c_1 = \alpha, c_2 = \alpha^3, c_3 = \alpha^7, c_4 = \alpha^{11}, c_5 = \alpha^{17}, c_6 = \alpha^5$. *The required* 3×6 *generator and the parity-check matrices for the* ℓ *constituent codes are as follows:*

$$
\begin{aligned}
\mathbf{G}_1 &= \left\langle \alpha^{16} \quad \alpha^{40} \quad \alpha^{41} \quad \alpha^{58} \quad \alpha^{19} \quad \alpha^{20} \right\rangle_3; \\
\mathbf{H}_1 &= \left\langle \alpha^{2} \quad \alpha^{5} \quad \alpha^{13} \quad \alpha^{23} \quad \alpha^{26} \quad \alpha^{34} \right\rangle_3 \\
\mathbf{G}_2 &= \left\langle \alpha^{8} \quad \alpha^{48} \quad \alpha \quad \alpha^{43} \quad \alpha^{29} \quad \alpha^{6} \right\rangle_3; \\
\mathbf{H}_2 &= \left\langle \alpha \quad \alpha^{6} \quad \alpha^{8} \quad \alpha^{29} \quad \alpha^{43} \quad \alpha^{48} \right\rangle_3 \\
\mathbf{G}_3 &= \left\langle \alpha^{24} \quad \alpha^{32} \quad \alpha^{3} \quad \alpha^{4} \quad \alpha^{53} \quad \alpha^{46} \right\rangle_3; \\
\mathbf{H}_3 &= \left\langle \alpha^{3} \quad \alpha^{4} \quad \alpha^{24} \quad \alpha^{32} \quad \alpha^{46} \quad \alpha^{53} \right\rangle_3 \\
\mathbf{G}_4 &= \left\langle \alpha^{17} \quad \alpha^{8} \quad \alpha \quad \alpha^{10} \quad \alpha^{53} \quad \alpha^{46} \right\rangle_3; \\
\mathbf{H}_4 &= \left\langle \alpha^{10} \quad \alpha \quad \alpha^{8} \quad \alpha^{17} \quad \alpha^{46} \quad \alpha^{53} \right\rangle_3 \\
\mathbf{G}_5 &= \left\langle \alpha^{2} \quad \alpha^{16} \quad \alpha^{33} \quad \alpha^{58} \quad \alpha^{12} \quad \alpha^{23} \right\rangle_3; \\
\mathbf{H}_5 &= \left\langle \alpha^{16} \quad \alpha^{2} \quad \alpha^{12} \quad \alpha^{23} \quad \alpha^{33} \quad \alpha^{58} \right\rangle_3 \\
\mathbf{G}_6 &= \left\langle \alpha^{26} \quad \alpha^{24} \quad \alpha^{48} \quad \alpha \quad \alpha^{13} \quad \alpha^{46} \right\rangle_3; \\
\mathbf{H}_6 &= \left\langle \alpha^{19} \quad \alpha^{3} \quad \alpha^{6} \quad \alpha^{8} \quad \alpha^{41} \quad \alpha^{53} \right\rangle_3.
\end{aligned}
$$

For $\beta_1 = 1$ and $\beta_2 = \alpha^{13}$, the calculated determinant values of $\mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\,\mathrm{T}}$ are displayed in Table 3.2. Further, for $\beta_0 = 1$ (when the constituent and their duals are *trivial* LCD MRD codes), the determinant value $\left|\mathbf{G}_{\mathrm{BM}}^{\beta_0}\mathbf{G}_{\mathrm{BM}}^{\beta_0\mathrm{T}}\right|$ is found to be $0; \mathbf{G}_{\mathrm{BM}}^{\beta_0}\mathbf{G}_{\mathrm{BM}}^{\beta_0\mathrm{T}} = \mathbf{0}_{3\ell}$.

Example 3.7.3 *Examples over* $\mathbb{F}_{2^8}$*: Let* $t = 1, k = 4$, *and* $n = 2k = 8$. *With* $\ell = 2,3,4,5,6,7,8$, *consider the following self-dual bases from* $\mathbb{F}_2$: $\mathbf{h}_1 = \left(\alpha^{66}, \alpha^7, \alpha^9, \alpha^{15}, \alpha^{30}, \alpha^{38}, \alpha^{98}, \alpha^{131}\right), \mathbf{h}_2 = \left(\alpha^{54}, \alpha^{10}, \alpha^{14}, \alpha^{27}, \alpha^{43}, \alpha^{86}, \alpha^{126}, \alpha^{137}\right), \mathbf{h}_3 = \left(\alpha^{43}, \alpha^5, \alpha^7, \alpha^{27}, \alpha^{30}, \alpha^{40}, \alpha^{157}, \alpha^{249}\right), \mathbf{h}_4 = \left(\alpha^{20}, \alpha^5, \alpha^{19}, \alpha^{27}, \alpha^{63}, \alpha^{65}, \alpha^{149}, \alpha^{234}\right), \mathbf{h}_5 = \left(\alpha^{15}, \alpha^7, \alpha^9, \alpha^{30}, \alpha^{38}, \alpha^{66}, \alpha^{98}, \alpha^{131}\right)$, $\mathbf{h}_6 = \left(\alpha^{10}, \alpha^7, \alpha^{14}, \alpha^{27}, \alpha^{43}, \alpha^{63}, \alpha^{244}, \alpha^{249}\right), \mathbf{h}_7 = \left(\alpha^9, \alpha^7, \alpha^{14}, \alpha^{28}, \alpha^{60}, \alpha^{76}, \alpha^{93}, \alpha^{252}\right)$, *and*

Table 3.2 Determinant Values of $G_{BM}^{\beta_1} G_{BM}^{\beta_1 T}$ and $G_{BM}^{\beta_2} G_{BM}^{\beta_2 T}$

ℓ	$\left\lvert \mathbf{G}_{BM}^{\beta_1} \mathbf{G}_{BM}^{\beta_1 T} \right\rvert$	$\left\lvert \mathbf{G}_{BM}^{\beta_2} \mathbf{G}_{BM}^{\beta_2 T} \right\rvert$
2	α^{28}	α^{15}
3	α^{42}	α^{48}
4	α^{7}	α^{41}
5	α^{35}	α^{38}
6	α^{49}	α^{10}

$\mathbf{h}_8 = \left(\alpha^5, \alpha^7, \alpha^{18}, \alpha^{19}, \alpha^{27}, \alpha^{63}, \alpha^{87}, \alpha^{249}\right)$, *where* $1 + \alpha^2 + \alpha^3 + \alpha^6 + \alpha^8 = 0$. *Let* $\mathbf{g}_i^{[0]} = c_i \mathbf{h}_i^{[4]}$ *for* $i = 1,2,3,4,5,6,7,8$ *with* $c_1 = \alpha, c_2 = \alpha^3, c_3 = \alpha^5, c_4 = \alpha^7, c_5 = \alpha^{11}, c_6 = \alpha^{13}, c_7 = \alpha^{23}$ $c_8 = \alpha^{29}$. The required 4×8 generator and the parity-check matrices of ℓ constituent codes are

$$
\begin{aligned}
\mathbf{G}_1 &= \left\langle \alpha^{36} \quad \alpha^{112} \quad \alpha^{144} \quad \alpha^{240} \quad \alpha^{225} \quad \alpha^{98} \quad \alpha^{38} \quad \alpha^{56} \right\rangle_4; \\
\mathbf{H}_1 &= \left\langle \alpha^{66} \quad \alpha^{7} \quad \alpha^{9} \quad \alpha^{15} \quad \alpha^{30} \quad \alpha^{38} \quad \alpha^{98} \quad \alpha^{131} \right\rangle_4 \\
\mathbf{G}_2 &= \left\langle \alpha^{99} \quad \alpha^{160} \quad \alpha^{224} \quad \alpha^{177} \quad \alpha^{178} \quad \alpha^{101} \quad \alpha^{231} \quad \alpha^{152} \right\rangle_4; \\
\mathbf{H}_2 &= \left\langle \alpha^{54} \quad \alpha^{10} \quad \alpha^{14} \quad \alpha^{27} \quad \alpha^{43} \quad \alpha^{86} \quad \alpha^{126} \quad \alpha^{137} \right\rangle_4 \\
\mathbf{G}_3 &= \left\langle \alpha^{178} \quad \alpha^{80} \quad \alpha^{112} \quad \alpha^{177} \quad \alpha^{225} \quad \alpha^{130} \quad \alpha^{217} \quad \alpha^{159} \right\rangle_4; \\
\mathbf{H}_3 &= \left\langle \alpha^{43} \quad \alpha^{5} \quad \alpha^{7} \quad \alpha^{27} \quad \alpha^{30} \quad \alpha^{40} \quad \alpha^{157} \quad \alpha^{249} \right\rangle_4 \\
\mathbf{G}_4 &= \left\langle \alpha^{65} \quad \alpha^{90} \quad \alpha^{49} \quad \alpha^{177} \quad \alpha^{243} \quad \alpha^{20} \quad \alpha^{89} \quad \alpha^{174} \right\rangle_4; \\
\mathbf{H}_4 &= \left\langle \alpha^{20} \quad \alpha^{5} \quad \alpha^{19} \quad \alpha^{27} \quad \alpha^{63} \quad \alpha^{65} \quad \alpha^{149} \quad \alpha^{234} \right\rangle_4 \\
\mathbf{G}_5 &= \left\langle \alpha^{240} \quad \alpha^{112} \quad \alpha^{144} \quad \alpha^{225} \quad \alpha^{98} \quad \alpha^{36} \quad \alpha^{38} \quad \alpha^{56} \right\rangle_4; \\
\mathbf{H}_5 &= \left\langle \alpha^{15} \quad \alpha^{7} \quad \alpha^{9} \quad \alpha^{30} \quad \alpha^{38} \quad \alpha^{66} \quad \alpha^{98} \quad \alpha^{131} \right\rangle_4 \\
\mathbf{G}_6 &= \left\langle \alpha^{160} \quad \alpha^{112} \quad \alpha^{224} \quad \alpha^{177} \quad \alpha^{178} \quad \alpha^{243} \quad \alpha^{79} \quad \alpha^{159} \right\rangle_4; \\
\mathbf{H}_6 &= \left\langle \alpha^{10} \quad \alpha^{7} \quad \alpha^{14} \quad \alpha^{27} \quad \alpha^{43} \quad \alpha^{63} \quad \alpha^{244} \quad \alpha^{249} \right\rangle_4 \\
\mathbf{G}_7 &= \left\langle \alpha^{144} \quad \alpha^{112} \quad \alpha^{224} \quad \alpha^{193} \quad \alpha^{195} \quad \alpha^{196} \quad \alpha^{213} \quad \alpha^{207} \right\rangle_4; \\
\mathbf{H}_7 &= \left\langle \alpha^{9} \quad \alpha^{7} \quad \alpha^{14} \quad \alpha^{28} \quad \alpha^{60} \quad \alpha^{76} \quad \alpha^{93} \quad \alpha^{252} \right\rangle_4 \\
\mathbf{G}_8 &= \left\langle \alpha^{80} \quad \alpha^{112} \quad \alpha^{33} \quad \alpha^{49} \quad \alpha^{177} \quad \alpha^{243} \quad \alpha^{117} \quad \alpha^{159} \right\rangle_4; \\
\mathbf{H}_8 &= \left\langle \alpha^{5} \quad \alpha^{7} \quad \alpha^{18} \quad \alpha^{19} \quad \alpha^{27} \quad \alpha^{63} \quad \alpha^{87} \quad \alpha^{249} \right\rangle_4 .
\end{aligned}
$$

Table 3.3 Determinant Values of $G_{BM}^{\beta_1} G_{BM}^{\beta_1 T}$ and $G_{BM}^{\beta_2} G_{BM}^{\beta_2 T}$

ℓ	$\left\lvert \mathbf{G}_{BM}^{\beta_1} \mathbf{G}_{BM}^{\beta_1 T} \right\rvert$	$\left\lvert \mathbf{G}_{BM}^{\beta_2} \mathbf{G}_{BM}^{\beta_2 T} \right\rvert$
2	α^{12}	α^{57}
3	α^{188}	α^{29}
4	α^{36}	α^{137}
5	α^{69}	α^{18}
6	α^{82}	α^{211}
7	α^{20}	α^{32}
8	α^{68}	α^{216}

The determinant values of $\left\lvert \mathbf{G}_{BM}^{\beta} \mathbf{G}_{BM}^{\beta\, T'} \right\rvert$ are given in Table 3.3 for $\beta_1 = 1$ and $\beta_2 = \alpha^{17}$. Further, for the case when the constituents and their duals are *trivial* LCD MRD codes, that is, $\beta = \beta_0$, the matrix $\mathbf{G}_{BM}^{\beta_0} \mathbf{G}_{BM}^{\beta_0 \mathbf{T}}$ is found to be singular for all $2 \le \ell \le 8$, and thus $\mathcal{C}_{BM}^{\beta_0}$ is not LCD. For $\ell = 2, 4, 6$, and 8, $\mathbf{G}_{BM}^{\beta_0} \mathbf{G}_{BM}^{\beta_0 \mathbf{T}} = \mathbf{0}_{4\ell}$ and thus the corresponding Block-Matrix code $\mathcal{C}_{BM}^{\beta_0}$ is not LCD.

3.7.2 Examples of q-Cyclic β-Construction: Construction-II

$$\mathbf{G}_{BM}^{q\beta} = \begin{pmatrix} \mathbf{G}^{(0)} & \mathcal{H}^{(1)} & \dots & \mathcal{H}^{(\ell-1)} \\ \mathcal{H}^{(0)} & \mathbf{G}^{(1)} & \dots & \mathcal{H}^{(\ell-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{H}^{(0)} & \mathcal{H}^{(1)} & \dots & \mathbf{G}^{(\ell-1)} \end{pmatrix} = \begin{pmatrix} \mathbf{G}^{(0)} & \beta\mathbf{H}^{(1)} & \dots & \beta\mathbf{H}^{(\ell-1)} \\ \beta\mathbf{H}^{(0)} & \mathbf{G}^{(1)} & \dots & \beta\mathbf{H}^{(\ell-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \beta\mathbf{H}^{(0)} & \beta\mathbf{H}^{(1)} & \dots & \mathbf{G}^{(\ell-1)} \end{pmatrix},$$

where $\mathbf{G}^{(i)} \mathbf{G}^{(i)^T} = \mathrm{diag}\left(\alpha_1^{[i]}, \alpha_1^{[i+1]}, \dots, \alpha_1^{[i+k-1]}\right)$ and $\mathcal{H}^{(i)} = \beta \mathbf{H}^{(i)}$ with $\alpha_1, \beta \in \mathbb{F}^*_{\left(2^t\right)^{2k}}$ such that β^2 is not a conjugate of α_1. In fact, the Block-Matrix LCD code $\mathcal{C}_{BM}^{q\beta}\left(n^{\times \ell}, k^{\times \ell}, d\right)$ is the generalization of the *self-dual* β construction $\mathcal{C}_{BM}\left(n^{\times \ell}, k^{\times \ell}, d\right)$ of Kumar and Durai (2021) for $\beta = 1$ and ℓ (number of constituent codes) is even. In this special construction, the Block-Matrix LCD code $\mathcal{C}_{BM}^{q\beta}\left(n^{\times \ell}, k^{\times \ell}, d\right)$ employs only a single *self-dual* basis from $\mathbb{F}_{\left(2^t\right)^{2k}}$. Following examples illustrate the q-cyclic β-construction of Block-Matrix codes in all the three cases $\beta = \beta_0, \beta_1$ and β_2.

Example 3.7.4 *Example with a self-dual basis over* $\mathbb{F}_{2^4}$

Let $t = 1, k = 2$, and $n = 2k = 4$. Consider a *self-dual* basis $\mathbf{h} = \left(\alpha^6, \alpha^{14}, \alpha^9, \alpha^{11}\right)$ of $\mathbb{F}_{2^4}$, where the primitive element α satisfies $1 + \alpha^3 + \alpha^4 = 0$. Then the $2\ell \times 4\ell$ matrix $\mathbf{G}_{BM}^{q\beta}$ over

$\mathbb{F}_{2^4}$ is composed of $\ell\, 2\times 4$ generator and ℓ parity-check matrices with the following initial matrices:

$$\mathbf{G}^{(0)} = \langle \alpha^8\ \alpha^{10}\ \alpha^5\ \alpha^{13}\rangle_2; \mathbf{H}^{(0)} = \langle \alpha^6\ \alpha^{14}\ \alpha^9\ \alpha^{11}\rangle_2,$$

where $\mathbf{g}^{[0]} = c\mathbf{h}^{[2]}$ with $c = \alpha^{14}$. The determinant values of $\left|\mathbf{G}_{\mathrm{BM}}^{q\beta}\mathbf{G}_{\mathrm{BM}}^{q\beta\mathrm{T}}\right|$ are given in Table 3.4 with $\beta_1 = 1$ and $\beta_2 = \alpha^3$. Also, for $\beta = \beta_0$, we have $\left|\mathbf{G}_{\mathrm{BM}}^{q\beta_0}\mathbf{G}_{\mathrm{BM}}^{q\beta_0\mathrm{T}}\right| = 0$; thus, $\mathcal{C}_{\mathrm{BM}}^{q\beta_0}$ is not an LCD code. Same is the case for $\ell = 2,4$, as $\mathbf{G}_{\mathrm{BM}}^{q\beta_0}\mathbf{G}_{\mathrm{BM}}^{q\beta_0\,\mathrm{T}} = 0_{2\ell}$.

Table 3.4 Determinant Values of $\mathrm{G}_{\mathrm{BM}}^{q\beta_1}\mathrm{G}_{\mathrm{BM}}^{q\beta_1\,\mathrm{T}}$ and $\mathrm{G}_{\mathrm{BM}}^{q\beta_2}\mathrm{G}_{\mathrm{BM}}^{q\beta_2\,\mathrm{T}}$

ℓ	$\left\|\mathbf{G}_{\mathrm{BM}}^{q\beta_1}\mathbf{G}_{\mathrm{BM}}^{q\beta_1\mathrm{T}}\right\|$	$\left\|\mathbf{G}_{\mathrm{BM}}^{q\beta_2}\mathbf{G}_{\mathrm{BM}}^{q\beta_2\mathrm{T}}\right\|$
2	α^9	α^{12}
3	α^{12}	α^{10}
4	1	α^8

Example 3.7.5 *Example with a self-dual basis over* $\mathbb{F}_{2^6}$*:* Let $t = 1, k = 3$, and $n = 2k = 6$. Consider the *self-dual* basis $\mathbf{h} = \left(\alpha^3, \alpha^4, \alpha^{33}, \alpha^{41}, \alpha^{52}, \alpha^{58}\right)$, with α being a primitive element in $\mathbb{F}_{2^6}$ satisfying $1 + \alpha^5 + \alpha^6 = 0$, so that the $3\ell \times 6\ell$ matrix $\mathbf{G}_{\mathrm{BM}}^{q\beta}$ over $\mathbb{F}_{2^6}$ is composed of $\ell\, 3\times 6$ generator and ℓ parity-check matrices with the following initial matrices:

$$G^{(0)} = \langle \alpha^{16} \quad \alpha^{24} \quad \alpha^4 \quad \alpha^5 \quad \alpha^{30} \quad \alpha^{15}\rangle_3;$$

$$H^{(0)} = \langle \alpha^3 \quad \alpha^4 \quad \alpha^{33} \quad \alpha^{41} \quad \alpha^{52} \quad \alpha^{58}\rangle_3,$$

where $\mathbf{g}^{[0]} = c\mathbf{h}^{[3]}$ with $c = \alpha^{55}$. Table 3.5 provides the determinant values of $\left|\mathbf{G}_{\mathrm{BM}}^{q\beta}\mathbf{G}_{\mathrm{BM}}^{q\beta\,\mathrm{T}}\right|$ for $\beta_1 = 1$ and $\beta_2 = \alpha^3$. As with $\beta = \beta_0, \left|\mathbf{G}_{\mathrm{BM}}^{q\beta_0}\mathbf{G}_{\mathrm{BM}}^{q\beta_0\,\mathrm{T}^2}\right| = 0$ for $2 \le \ell \le 6, \mathcal{C}_{\mathrm{BM}}^{q\beta_0}$ is not an LCD code. Also, for $\ell = 2,4,6$, we have $\mathbf{G}_{\mathrm{BM}}^{q\beta_0}\mathbf{G}_{\mathrm{BM}}^{q\beta_0\,\mathrm{T}} = 0$, and hence $\mathcal{C}_{\mathrm{BM}}^{q\beta_0}$ is not an LCD code.

Table 3.5 Determinant Values of $\mathrm{G}_{\mathrm{BM}}^{q\beta_1}\mathrm{G}_{\mathrm{BM}}^{q\beta_1\mathrm{T}}$ and $\mathrm{G}_{\mathrm{BM}}^{q\beta_2}\mathrm{G}_{\mathrm{BM}}^{q\beta_2\mathrm{T}}$

ℓ	$\left\|\mathbf{G}_{\mathrm{BM}}^{q\beta_1}\mathbf{G}_{\mathrm{BM}}^{q\beta_1\mathrm{T}}\right\|$	$\left\|\mathbf{G}_{\mathrm{BM}}^{q\beta_2}\mathbf{G}_{\mathrm{BM}}^{q\beta_2\mathrm{T}}\right\|$
2	1	α^{24}
3	α^{56}	α^{10}
4	1	α^{19}
5	α^{42}	α^{32}
6	1	α^{12}

Example 3.7.6 *Example with a self-dual basis over* $\mathbb{F}_2$*:* Let $t=1$, $k=4$ and $n=2k=8$. Consider the *self-dual* basis $\mathbf{h}=(\alpha^{32},\alpha^{64},\alpha^{240},\alpha,\alpha^{129},\alpha^{195},\alpha^{69},\alpha^{111})$ and the following 3×6 generator and parity-check matrices, with α being a primitive element in $\mathbb{F}_{2^6}$ satisfying $1+\alpha^5+\alpha^6=0$. Then the $4\ell\times 8\ell$ matrix $\mathbf{G}_{\mathrm{BM}}^{q\beta}$ over $\mathbb{F}_{2^8}$ is composed of ℓ 4×8 generator and ℓ parity-check matrices with the following initial matrices:

$$\mathbf{G}^{(0)}=\langle \alpha^{32} \quad \alpha^{64} \quad \alpha^{240} \quad \alpha \quad \alpha^{129} \quad \alpha^{195} \quad \alpha^{69} \quad \alpha^{111}\rangle_4;$$

$$\mathbf{H}^{(0)}=\langle \alpha^{5} \quad \alpha^{7} \quad \alpha^{18} \quad \alpha^{19} \quad \alpha^{27} \quad \alpha^{63} \quad \alpha^{87} \quad \alpha^{249}\rangle_4$$

where $\mathbf{g}^{[0]}=c\mathbf{h}^{[3]}$ with $c=\alpha^{55}$. Table 3.6 presents the determinant values of $\left|\mathbf{G}_{\mathrm{BM}}^{q\beta}\mathbf{G}_{\mathrm{BM}}^{q\beta\,\mathrm{T}}\right|$ for $\beta_1=1$ and $\beta_2=\alpha^3$. Also, with , $\beta=\beta_0$, $\left|\mathbf{G}_{\mathrm{BM}}^{q\beta_0}\mathbf{G}_{\mathrm{BM}}^{q\beta_0\,\mathrm{T}}\right|=0$ for all $2\le\ell\le 6$; thus, $\mathcal{C}_{\mathrm{BM}}^{q\beta_0}$ is not an LCD code. Also, for $\ell=2,4,6$ and 8, we have $\mathbf{G}_{\mathrm{BM}}^{q\beta_0}\mathbf{G}_{\mathrm{BM}}^{q\beta_0\mathrm{T}}=0$, and hence $\mathcal{C}_{\mathrm{BM}}^{q\beta_0}$ is not LCD.

Table 3.6 Determinant Values of $\mathrm{G}_{\mathrm{BM}}^{q\beta_1}\mathrm{G}_{\mathrm{BM}}^{q\beta_1\,\mathrm{T}}$ and $\mathrm{G}_{\mathrm{BM}}^{q\beta_2}\mathrm{G}_{\mathrm{BM}}^{q\beta_2\,\mathrm{T}}$

ℓ	$\left\|\mathbf{G}_{\mathrm{BM}}^{q\beta_1}\mathbf{G}_{\mathrm{BM}}^{q\beta_1\mathrm{T}}\right\|$	$\left\|\mathbf{G}_{\mathrm{BM}}^{q\beta_2}\mathbf{G}_{\mathrm{BM}}^{q\beta_2\mathrm{T}}\right\|$
2	α^{165}	α^{5}
3	α^{225}	α^{225}
4	α^{60}	α^{114}
5	α^{15}	α^{15}
6	α^{150}	α^{215}
7	α^{30}	α^{30}
8	1	α^{212}

3.8 CONCLUSION

Constructions of LCD codes of length larger than n are considered in this work. For a given $\ell>0$, construction of LCD codes are of length $n\ell$ over $\mathbb{F}_{\left(2^t\right)^{2k}}$ are proposed from ℓ LCD MRD codes: consideration is given to various cases when the constituent codes and their duals are *trivial* and *non-trivial* LCD codes. A particular class of Block-Matrix codes is discussed in which the constituent codes are constructed using 2^t-cycles of a given *self-dual*

basis of $\mathbb{F}_{\left(2^t\right)^{2k}}$ with characteristic 2. In general, the constructed codes—termed Block-Matrix LCD codes $\mathcal{C}_{\mathrm{BM}}\left(n^{\times\ell}, k^{\times\ell}, d\right)_{\left(2^t\right)^{2k}}$- can employ at most ℓ *self-dual* bases to construct *non-trivial* (or *trivial*) LCD MRD codes $\left(\mathbf{G}_i\mathbf{G}_i{}^{\mathrm{T}} \neq \mathbf{I}_k\right)$ and their corresponding *trivial* (or *non-trivial*) LCD MRD duals $\left(\mathbf{H}_i\mathbf{H}_i{}^{\mathrm{T}} \neq \mathbf{I}_k\right)$. A direct formula to compute the determinant $\left|\mathbf{G}_{\mathrm{BM}}\mathbf{G}_{\mathrm{BM}}^{\mathrm{T}}\right|$ is obtained to justify the complementary dual property of the proposed Block-Matrix codes. The necessary and sufficient condition under which the $k\ell \times k\ell$ matrix $\mathbf{G}_{\mathrm{BM}}^{\beta}\mathbf{G}_{\mathrm{BM}}^{\beta\,\mathrm{T}}$ is non-singular is given. Several illustrations are provided in support of the proposed code constructions.

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Chapter 4

Challenges in Protecting Multimedia Data

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4.1 INTRODUCTION

In today's digital landscape, multimedia content such as images, audio, and video permeates networks, social media, and applications. Video traffic encompasses various sectors, including entertainment, the military, and healthcare. This widespread presence elevates the importance of security; ensuring the confidentiality, integrity, and authenticity of multimedia materials is essential. Encryption is vital for protecting content by guaranteeing confidentiality, authentication, integrity, and non-repudiation. However, evolving threats and different types of data present significant challenges. According to Alomoush et al. (2023), safeguarding multimedia requires addressing several issues: preventing unauthorized access, tracking the origin of leaks, stopping unauthorized alterations, and protecting intellectual property. In practice, this involves implementing multilayered security measures, such as encryption, watermarking, and steganography, throughout storage, transmission, and usage. Additionally, modern security strategies must consider emerging threats (e.g., generative artificial intelligence (AI) fakes) and the operational limits of real-time media (e.g., streaming capabilities, Internet of Things (IoT) devices) (Alomoush et al., 2023).

Historically, information security treated text data and structured data with classical methods of encryption, authentication, and access control. Multimedia data is different by nature; its unstructured, large-sized, and multiformatted nature requires protection strategies different from those of static text (Nisa & Baba, 2024). Unlike text, multimedia is easily copied, altered, or forged without any traces being readily detectable, which complicates its authentication or source verification. In addition, its dense sensory character permits the attacker to leverage emotional influence, such as in the abuse of doctored video, deceptive audio recording, or pilfered imagery in disinformation operations. With the increasing size and sophistication of multimedia systems, especially with the mainstream adoption of cloud computing, mobile communications, and immersive technologies like virtual and augmented reality (AR), the necessity for strong and resilient protection mechanisms has never been greater (Yang et al., 2022).

The core objective of this chapter is to provide a comprehensive analysis of the contemporary challenges in protecting multimedia data. The aim of this chapter is to provide an in-depth evaluation of current challenges, technical gaps, and contemporary solutions in multimedia data protection. It also examines user privacy, legal compliance, ethical considerations, and the integration of forensic tools for tamper detection and evidence preservation. Additionally, compliance with legal and ethical standards such as the General Data Protection Regulation (GDPR), California Consumer Privacy Act (CCPA), and intellectual property laws must be balanced against usability, openness, and innovation. Within this scope, the

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chapter explores key security domains including encryption, watermarking, steganography, content filtering, blockchain-enabled provenance, and digital rights management (DRM). It also analyzes application-specific challenges in cloud-based content storage, mobile platforms, social media sharing, and immersive technologies like AR and virtual reality (VR). Specific issues such as content identification and filtering, secure multimedia storage and cloud distribution, protection of user-generated content (UGC) on social media, and the safeguarding of immersive formats are also examined in detail. In addition, this chapter investigates the role of user authentication, access control, and forensic analysis in verifying the authenticity and integrity of multimedia content. These concerns are not only technical but also ethical and legal, requiring alignment with regulatory frameworks such as the GDPR and other global data protection standards.

A central problem this chapter addresses is the inadequacy of legacy security infrastructures to handle the dynamic and context-sensitive nature of multimedia. Techniques that work well for text-based data, such as simple hashing or static encryption, may fail to account for the fluid, real-time nature of video streaming or the context-driven interpretation of visual content. Another challenge lies in the scalability of protection methods. Platforms like YouTube, Instagram, and TikTok host billions of pieces of content, demanding real-time monitoring and filtering without compromising performance or user experience. The chapter also highlights the limitations of current DRM systems, which often prioritize content restriction over flexibility, sometimes to the detriment of legitimate users and creators.

The protection of multimedia data is not just a technical necessity; it is a foundation for maintaining digital trust, protecting intellectual property, and safeguarding individual and collective identities. In a world where information is increasingly visual and experiential, ensuring the authenticity, confidentiality, and proper attribution of content is vital for social, political, and economic stability. As AI-generated media becomes more realistic and harder to detect, and as multimedia platforms become more decentralized and complex, stakeholders must rethink security strategies. Solutions must be proactive, multilayered, and adaptable, incorporating advances in machine learning (ML), cryptography, blockchain, and privacy engineering.

To structure this investigation, the chapter proceeds through several interconnected topics. It begins with a discussion on content identification and filtering, exploring the algorithms and systems used to detect copyrighted or inappropriate material. It then examines cloud storage and distribution, focusing on how encryption, compliance, and key management intersect in multimedia contexts. The following sections address the security challenges inherent in social media sharing and the adaptation of protection strategies for emerging technologies like AR/VR. User authentication and access control are explored next, followed by a detailed discussion on multimedia forensic analysis.

4.1.1 Main Challenges in Protecting Multimedia Data

- Preventing unauthorized disclosure: Ensuring that only authorized parties can access sensitive multimedia content.
- Leak detection and source tracing: Embedding mechanisms to trace or identify sources of illicit redistribution.
- Integrity and tamper resistance: Guaranteeing that media are not altered without authorization (e.g., no hidden forgery or edit).
- Intellectual property protection: Embedding proofs of ownership so that copyright can be enforced.

- Performance and capacity limits: Balancing protection with bandwidth, latency, and device constraints (e.g., live video encryption needs efficiency, and IoT systems demand lightweight methods).
- Evolving attacks: Adversaries now use sophisticated tools (e.g., deepfake generators) to fool security measures, requiring robust countermeasures.

Objectives of the Chapter

The objectives of the chapter are as follows:

- To explore the challenges in securing multimedia data across digital platforms, examining vulnerabilities in storage, transmission, and access control;
- To provide a comprehensive analysis of contemporary challenges in protecting multimedia data across various digital platforms;
- To evaluate current technical gaps and solutions for multimedia data protection, including user privacy, legal compliance, and ethical considerations;
- And, to examine the integration of forensic tools for tamper detection and evidence preservation in multimedia security frameworks.

Organization of the Chapter

The rest of the chapter is organized as follows. Section 4.2 illustrates diverse threats to multimedia data. Section 4.3 discusses encryption and secure transmission. Section 4.4 discusses digital watermarking and data hiding. Section 4.5 discusses steganography and covert channels. Section 4.6 discusses the authentication and integrity. Section 4.7 discusses DRM systems. Section 4.8 discusses the blockchain and distributed security. Section 4.9 discusses content identification and filtering. Section 4.10 discusses cloud storage and distribution. Section 4.11 discusses the sharing of social media. Section 4.12 discusses emerging technologies, such as AR, VR, blockchain technologies, non-fungible tokens, ML, cloud/edge/quantum computing, natural language processing (NLP), and AI. Section 4.13 discusses the user authentication and access control. Section 4.14 discusses forensic analysis, in which reliable techniques are developed for detecting tampering and verifying the authenticity of multimedia content. And, finally, Section 4.15 concludes the chapter.

4.2 THREATS TO MULTIMEDIA DATA

Multimedia data are susceptible to standard threats concerning confidentiality, integrity, and availability, although with distinctive nuances. Attackers may intercept videos or audio streams, manipulate content, or extract or remove concealed data. For instance, media files are subject to forgery or manipulation using techniques such as image splicing or audio synthesis. The advent of deep learning has rendered realistic forgeries, commonly referred to as "deepfakes," increasingly prevalent. In the domain of image forensics, copy-move and splicing attacks are prevalent; an attacker may replicate a section of an image or substitute a segment with content from a different image. A survey conducted by Zanardelli et al. (2023) indicated that modern forgery detection systems must counteract copy-move and splicing attacks, in addition to effectively identifying deepfake image content, and further asserted that deep-learning techniques presently provide the highest accuracy in detection. In audio, threats to authentication and copyright arise from voice cloning and synthetic speech attacks. Recent research indicates that

existing audio watermarking schemes lack robustness against all adversarial distortions, implying that even protected audio may be susceptible to removal or modification through signal transformations or generative AI. Additional threats encompass traditional piracy, characterized by the unlawful duplication and distribution of media files, as well as insider leaks.

To address all these threat vectors, solutions must guarantee the following:

Confidentiality: Unauthorized individuals are precluded from accessing sensitive media (e.g., unauthorized listeners are unable to access encrypted video feeds).
Integrity: Any tampering, be it compression, cropping, or editing, is either prevented or detected (e.g., through checksums, digital signatures, or watermarks).
Authenticity and non-repudiation: The identity of content owners or creators is verifiable (e.g., through digital fingerprints or watermarking).
Availability: Access controls and redundancy ensure that media remain accessible to legitimate users, even in the face of an attack (e.g., providing resistance to denial-of-service attacks).

The variety of multimedia data encompassing raw versus compressed, streaming versus static, and high-volume versus small snippets indicates that no single technique is adequate. Contemporary research highlights the importance of multi-layer defense strategies; for instance, Gao and Wu (2024) contended that the combination of watermarking, encryption, and steganography yields a more resilient security framework. Watermarks or fingerprints serve to authenticate content and monitor distribution, encryption conceals the content, and steganography hides the presence of data collectively, which can counter threats such as deepfake generation and unlawful copying. The remainder of this chapter elaborates on these methods, the challenges they face, and their interconnections.

4.3 ENCRYPTION AND SECURE TRANSMISSION

Role of encryption: Encryption constitutes a fundamental instrument for ensuring the confidentiality of multimedia. Symmetric cryptography (AES) and asymmetric schemes (RSA, ECC) are capable of scrambling image, audio, or video data, ensuring that only individuals possessing the appropriate keys can decrypt it. Hosny et al. (2023) assert that, in contemporary communications, "encryption is vital . . . ensuring confidentiality, authentication, integrity, and non-repudiation" of multimedia content. Multimedia-specific encryption schemes might selectively apply these ciphers (e.g., encrypting only certain segments of video to expedite processing) or integrate chaos-based pseudorandom number generators to enhance complexity (Hosny et al., 2023)

4.3.1 Encryption Challenges for Multimedia

4.3.1.1 Computational Overhead

High-definition video streams are substantial, and fully encrypting every frame with a robust cipher can incur unacceptable latency. For live streaming or IoT cameras, encryption must be lightweight. Recent studies (e.g., Shafiq et al., 2024) proposed hybrid symmetric–asymmetric encryption (which combines a modified variant of the Advanced Encryption Standard (AES) with elliptic-curve cryptography) to accelerate processing while maintaining security. The results indicated an improved processing time compared with standard AES or ECC deployed in isolation.

4.3.1.2 Data Integrity in Transit

Encryption alone does not preempt the deletion or rearrangement of encrypted packets. Secure protocols (such as the Secure Real-Time Transport Protocol (SRTP) for real-time media) supplement encryption with integrity checks (Hash-Based Message Authentication Code) to detect packet tampering.

4.3.1.3 Key Management

The distribution and management of encryption keys at scale, particularly for broadcast or multicast multimedia, present a complex challenge. To prevent man-in-the-middle attacks, strong key-exchange and authentication protocols (such as Transport Layer Security (TLS) and Datagram Transport Layer Security) are essential.

4.3.1.4 Compression Compatibility

Multimedia is often compressed (e.g., H.264 video and JPEG images). Encryption may disrupt compression efficiency or synchronization. Some schemes implement encryption subsequent to compression; however, if the bitstream format incorporates markers (such as Group of Pictures boundaries), selective encryption may be employed (where only intra-coded frames or headers are encrypted) to preserve stream structure. Such selective or format-aware encryption can mitigate data overhead, though it may compromise security if attackers can deduce unencrypted portions.

Essentially, while encryption is indispensable, it should be optimized for multimedia formats. Numerous schemes specifically cater to images, videos, and audio. For instance, certain methods utilize chaotic maps to encrypt pixel blocks or rearrange pixels prior to encryption to enhance confusion. Moreover, encryption can be integrated with watermarking (refer to the subsequent section) to provide both confidentiality and ownership protection within a singular process, as noted by Hosny et al. (2023).

4.4 DIGITAL WATERMARKING AND DATA HIDING

Digital watermarking involves embedding imperceptible information (a "watermark") into multimedia content to assert ownership or verify integrity. This concealed signature is designed to endure benign transformations (such as compression or scaling) yet remain detectable by authorized algorithms. For example, Mekhfioui et al. (2025) defined watermarking as the practice of embedding a hidden payload in an image (typically within a transform domain) to safeguard digital assets. Authors emphasized that watermarking has emerged as an "essential solution for protecting digital content" in the contemporary landscape of online distribution. Watermarks may be categorized as robust (resilient to attacks) or fragile (breaking upon any modification), based on the intended goal. Robust watermarks serve to authenticate ownership and may withstand lossy compression or minor editing; conversely, fragile watermarks are tailored to detect any instances of tampering (Kaur et al., 2020; Mekhfioui et al., 2025).

4.4.1 Contemporary Trends and Challenges in Digital Watermarking

4.4.1.1 Applications

Watermarks can incorporate copyright information (such as owner identification), usage rights, timestamps, or concealed keys. In a video, an invisible watermark may be distributed



Figure 4.1 Example of invisible video watermarking.

Source: Bitmovin (2021)

frame-by-frame or across multiple frames. Audio watermarking similarly conceals bits within the frequency spectrum or the least-significant bits of samples. The Association for Computing Machinery (ACM) Multimedia 2023 challenge underscores invisible video watermarking as an effective method for imperceptibly tracking ownership (Chen et al., 2023).

4.4.1.2 Robustness versus Capacity

The amount of data that can be embedded, its visibility, and its resistance to assaults is all subject to a fundamental trade-off. Mekhfioui et al. (2025) affirmed that no watermarking technique guarantees complete protection under all conditions; each technique is customized to address specific media types and threat models. For instance, highly robust watermarks may become more perceptible or possess limited capacity.

4.4.1.3 Evolving Techniques

Initial methods embedded watermarks within spatial or transform domains (Discrete Cosine Transform (DCT), Discrete Wavelet Transform (DWT)). Contemporary approaches are increasingly leveraging deep neural networks. As Luo et al. (2024) noted, deep learning has significantly advanced watermarking methodologies by training encoder–decoder networks to identify resilient embedding spaces. These models can adaptively conceal a watermark in intricate ways resistant to conventional removal attacks. For example, convolutional autoencoders can effectively embed bits while preserving image quality. In turn, AI-based attacks (including removal networks or generative adversarial networks (GANs)) challenge these schemes, prompting the necessity for adversarial training (Luo et al., 2024). Figure 4.3 shows the architecture of a deep neural network used in a watermarking system, as described by Kim et al. (2021), which includes a feature extractor, encoder, and decoder for watermark embedding and detection.

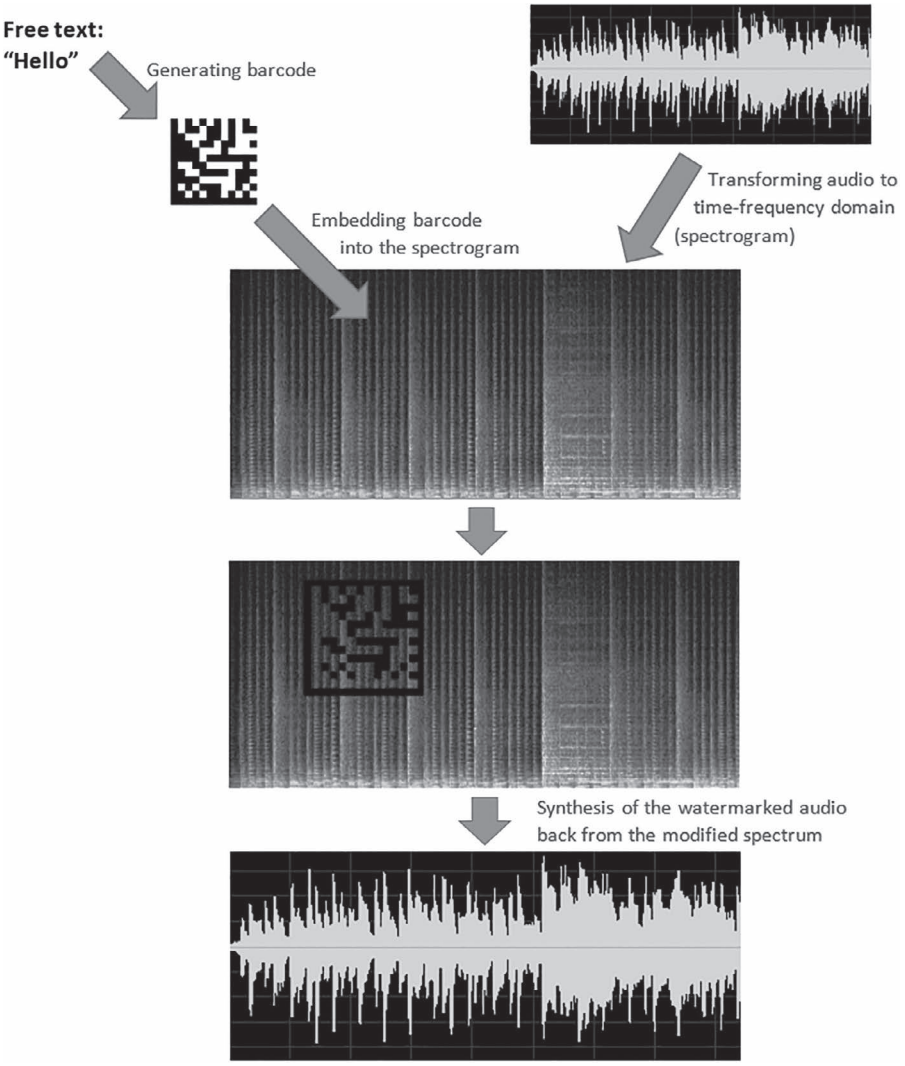


Figure 4.2 Spectrogram illustrating audio watermark embedding.

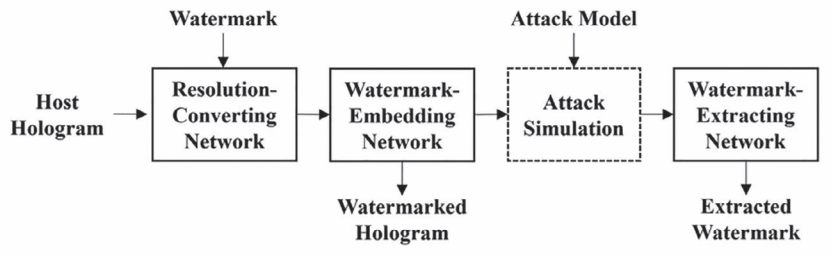


Figure 4.3 Structure of a deep neural network for watermark embedding and detection.

Source: Kim et al. (2021)

4.4.1.4 Watermark Detection

Depending on whether the original media is requisite, schemes are categorized as non-blind (where original content is required), semi-blind (only the watermark or key is necessary), or blind (where only the secret key is required). In numerous copyright disputes, blind extraction is preferred, as verification of ownership may need to occur without access to the original file. As shown in Figure 4.4, the proposed architecture by Chan et al. (2004) used both spatial and frequency domain embedding to increase resilience during detection.

4.4.1.5 Limitations

Wen et al. (2025) report that current audio watermarking methods, and many digital watermarking systems, in general, "are not robust enough to withstand all tested distortions in practice." Specifically, adversarial attacks (including those leveraging AI) can frequently remove or alter a watermark. This highlights an ongoing challenge: the design of watermarks capable of surviving modern attacks (e.g., neural adversarial transformations) while remaining imperceptible.

The Figure 4.4 below, from Chekira et al. (2023), illustrates the waveform comparison between original and watermarked audio samples across several genres. While the differences appear minimal to the naked eye, the signal alteration demonstrates the delicate balance between imperceptibility and robustness in audio watermarking.

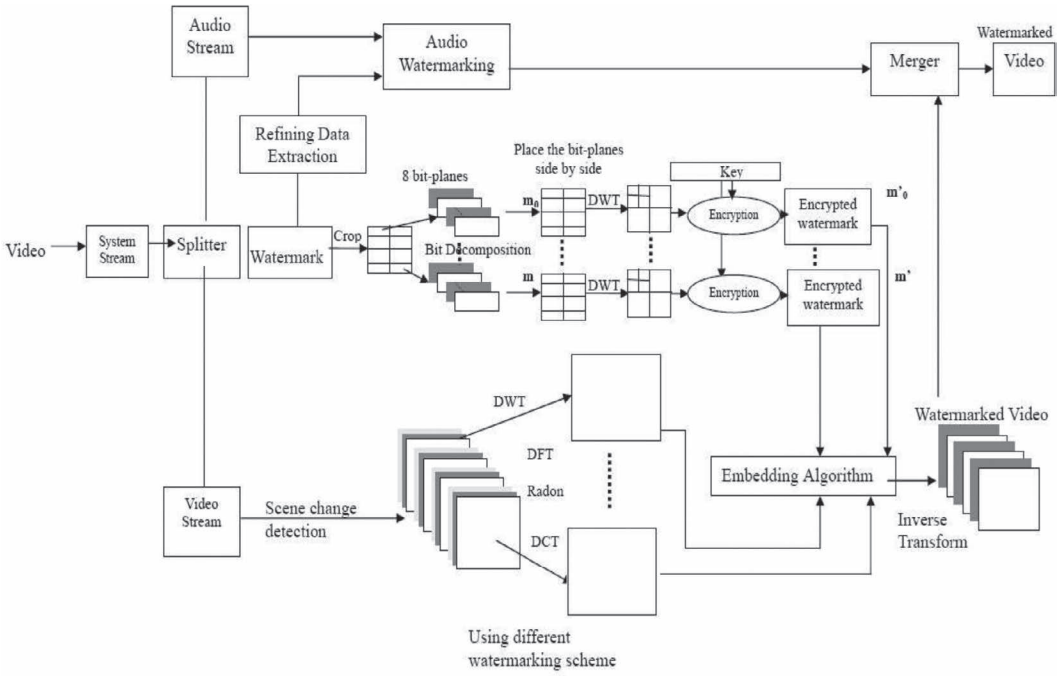


Figure 4.4 Architecture of hybrid video watermarking for robust detection.

Source: Adapted from Chan et al. (2004)

4.4.1.6 Future Directions

Researchers are advocating for hybrid schemes and multilayered watermarks. For instance, Mekhfioui et al. (2025) utilized multi-objective optimization (through shearlet transforms and deep learning) to enhance image watermark robustness under IoT constraints. Others propose the combination of fragile and robust marks to both detect tampering and validate origin.

Watermarking serves as a cornerstone for establishing ownership and identifying tampering in multimedia. When designed effectively, it complements encryption by providing traceability. Nevertheless, as current studies demonstrate, the simultaneous achievement of imperceptibility, capacity, and robustness remains a significant challenge.

4.5 STEGANOGRAPHY AND COVERT CHANNELS

Steganography conceals the *existence* of a message within multimedia data. Unlike encryption (which makes data unreadable) or watermarking (which authenticates content), steganography hides secret data payloads inside cover images/audio so that observers are unaware of any hidden content. Ghoul et al. (2023) explained that steganography is one promising tool for secure exchange of sensitive information over unsecured media, as hidden messages appear "undetectable to human eyes" when embedded in images or audio.

4.5.1 Important Considerations

- Capacities and trade-offs: A steganographic system aims for high embedding capacity (how much can be hidden), low perceptibility (no visible artifacts), and security against detection (steganalysis). These goals conflict: improving capacity or robustness often makes the payload easier to detect. Ghoul et al. note the trade-off explicitly: *enhancing one factor typically undermines one or more others*.
- Techniques: Early methods modify the least-significant bits (LSBs) of image pixels. Modern steganography uses transform-domain techniques (modifying DCT coefficients in JPEG, wavelet coefficients, etc.) and even neural networks (autoencoders) to embed data adaptively. Spread-spectrum, matrix embedding, and dithering are used to minimize distortion. Video steganography might embed data frame-by-frame or across keyframes to achieve higher throughput.
- Detection (steganalysis): Because steganography by design resists casual detection, analysts use statistical and ML-based detectors to find anomalies. For example, convolutional neural networks (CNNs) can detect patterns in pixel noise introduced by hiding bits. Good steganography aims to mimic cover data statistics so that detectors fail. However, as Ghoul et al. (2023) observed, comprehensive evaluation is lacking: many steganographic claims focus only on capacity or imperceptibility, ignoring advanced ML steganalysis.
- Applications: Steganography can securely send encryption keys or private messages hidden in innocent media (e.g., sending a covert key to a colleague embedded in a public video). It is also used in digital forensics for covert watermarking; for example, investigators might embed tracking codes in distributed copies of media. However, malicious actors also use steganography for covert communication (e.g., hiding instructions in images), so defenses must consider detection and policy.

In practice, steganography is often paired with encryption: first, encrypt the message, and then hide it in an image or audio. This provides confidentiality *and* deniability (an eavesdropper does not know a secret exists). However, as Ghoul et al. (2023) caution, standard steganography

tools have vulnerabilities, especially against deep-learning steganalysis. Ensuring "high security" (undetectability) remains an active research area. Overall, steganography is a useful supplement for multimedia protection (especially covert messaging), but it addresses a different goal than encryption or watermarking.

4.6 AUTHENTICATION AND INTEGRITY

Beyond hiding or encrypting content, multimedia systems need methods to verify integrity and authenticate the source. Common tools include the following:

- Cryptographic hashes and digital signatures: Compute a hash of the media (e.g., SHA-256) and sign it using public-key cryptography. Any alteration changes the hash and invalidates the signature. This is straightforward but requires managing keys and storing signatures (e.g., as metadata).
- Watermarking/fingerprinting: As mentioned, a robust watermark can serve as an implicit signature: detecting the correct watermark proves authenticity. The difference is that watermarks stay with the content, whereas digital signatures are separate. Fingerprinting is related: unique identifiers (e.g., user IDs) are embedded to trace distribution.
- Secure protocols: Transport mechanisms like SRTP combine encryption and integrity checks for media streams, ensuring packets can't be dropped or tampered with undetected.
- Perceptual hashes: For some applications, content is "hashed" in a perceptual way (e.g., image fingerprinting) so that similar content maps to similar hashes. This can detect unauthorized copies even if the content has minor edits.

Implementing these mechanisms at scale with multimedia streams is nontrivial. Signatures add overhead, and watermarking requires specialized detection tools. The design must consider the intended use: forensic tracing might tolerate heavier checks than live streaming.

4.7 DRM SYSTEMS

DRM systems are frameworks for controlling access and usage of multimedia content, especially copyrighted media. A DRM system typically encrypts media and ties decryption to license conditions (e.g., via a key server). Petrică and Arghirescu (2024) reviewed DRM's role in multimedia protection, emphasizing that DRM "restricts access to multimedia files or digital services with customized access." In practice, DRM solutions (e.g., PlayReady, Widevine) combine encryption with access control: only authorized devices with valid licenses can decrypt the media. DRM protects IP by making unauthorized copying useless without keys.

The following are some of the key points to consider:

- License enforcement: DRM ties content keys to licenses, which can specify constraints (e.g., no copying, limited playback count).
- Persistent protection: Even after decryption, some DRM schemes embed usage logs or forensic watermarks on playback to deter repeated illicit use.
- Interoperability issues: Different platforms use different DRM, which can hinder legitimate use. Standardization efforts (encrypted media extensions in browsers) aim to harmonize.
- Limitations: DRM can be unpopular (restricts consumer rights) and is only as strong as the key management. Attackers sometimes find device vulnerabilities to extract keys or remove DRM. It also doesn't protect content post-decryption (e.g., analog copying).

DRM reflects a policy layer on top of encryption and watermarking. From a technical standpoint, it leverages these tools to enforce usage rules. Petrică and Arghirescu (2024) noted, DRM is one of "several technical solutions for encrypting digital information," with particular emphasis on controlling multimedia access.

4.8 BLOCKCHAIN AND DISTRIBUTED SECURITY

Recently, blockchain and distributed ledger technologies had been explored for multimedia security. The immutability and decentralization of blockchain can help with provenance and transparency. For example, metadata about content origin (owner identity, creation timestamp) can be recorded on-chain, ensuring a tamper-evident log. Qureshi and Megías (2021) surveyed blockchain applications in multimedia protection, noting a "holistic survey of multimedia content protection applications in which blockchain technology is being used." Authors proposed a taxonomy that combines blockchain features (e.g., smart contracts) with content protection techniques (encryption, watermarking).

4.8.1 Benefits and Uses

- Proof of ownership: Registering content fingerprints on a blockchain lets anyone verify ownership. Each time content is shared, transactions can timestamp the event.
- Licensing and rights: Smart contracts can automate licensing rules (release keys upon payment).
- Transparent tracking: Content distribution (e.g., video views) could be logged, aiding royalty accounting.

Challenges include scalability (blockchains are slow and not efficient for large data) and privacy (public ledgers reveal usage patterns unless careful design). Nevertheless, blockchain is an active research area for DRM-like functions and audit trails in multimedia security.

4.8.2 Specialized Contexts: IoT, Cloud, and Mobile

Multimedia today often originates and resides on resource-constrained devices (IoT cameras, mobile phones) or in cloud environments:

- IoT and edge devices: Cameras and sensors in IoT capture images/video that may need protection. These devices often lack computing power. Mekhfioui et al. (2025) addressed this by optimizing watermarking for embedded systems with limited memory/CPU. Similarly, lightweight encryption schemes (e.g., chaos-based ciphers) are studied for IoT video streams. Nisa and Baba (2024) highlighted that multimedia in IoT raises unique issues: privacy and security of pervasive data, and they survey both traditional methods (encryption, watermarking) and emerging ones (blockchain, federated learning) for IoT content protection.
- Cloud and streaming: In cloud storage, multimedia must be encrypted at rest, and cloud services often implement DRM. End-to-end encryption is a challenge for multi-tenant clouds (trusted execution environments or client-side encryption may be used). For streaming, protocols like HTTPS and HLS with DRM ensure secure delivery.
- Mobile platforms: Protecting UGC (photos, voice notes) on mobile devices involves OS-level DRM (e.g., secure keys in a trusted execution environment) and permissions. Also, some mobile apps apply perceptual encryption (like blurring faces for privacy) or are

investigating privacy-preserving computation (e.g., homomorphic encryption, secure multi-party computation for processing sensitive images without revealing them fully).

In each domain, trade-offs appear; more protection can mean less performance or usability. For example, highly secure watermarking might increase battery usage, so research seeks efficient algorithms (e.g., real-time, autoencoder-based watermarking optimized for ARM processors).

4.8.3 Emerging Threats: AI and Deepfakes

A particularly hot challenge is multimedia forged by generative AI. Deepfake images and voices can mimic real people, compromising trust. Traditional protections struggle here: an attacker can generate a "fake" image or audio that is visually plausible and also free of original watermarks. Gao and Wu (2024) remarked that modern security techniques must withstand "threats such as deepfake manipulations [and] AI-based attacks."

Possible defenses being explored include:

- Robust watermarking: Embedding watermarks before media generation, or in pre-trained content, so that deepfake outputs (which copy from a source) inherit a traceable watermark. The 2023 Invisible Watermark challenge aims to test watermarking under GAN synthesis.
- Active forgery detection: Designing algorithms (often deep neural networks) that detect inconsistencies or anomalies in AI-generated content. For example, analyzing physical plausibility (lighting, shadows) or temporal coherence in video.
- Content provenance frameworks: Combining blockchain and watermark logs to verify the authenticity chain, that is, check if a video's blockchain record exists from the original capture.

Wen et al.'s (2025) evaluation of audio watermarking finds that *none of the current schemes is robust to all AI-driven distortions*. This suggests a need for new approaches. Ongoing research includes adversarial training (watermarks designed to resist GAN attacks) and hybrid approaches (fractal watermarks, optical content patterns). This is a rapidly evolving front in multimedia security.

4.8.4 Performance and Quality Trade-Offs

Throughout all protective techniques, performance impacts and quality trade-offs are critical:

- Computational cost: Encryption and watermarking add overhead. Researchers often measure PSNR (for image/video quality) and processing time. A practical system must keep PSNR high (imperceptibility) and latency low.
- Compression effects: Strong compression can weaken or destroy hidden data. Methods must account for typical compression pipelines (JPEG, H.264) used in Internet media.
- Bandwidth constraints: For live streams, added metadata (e.g., digital signatures or watermark payload) should not inflate the size greatly.
- Synchronization: In streaming multimedia, aligning keys or verifying watermarks must be real-time or near-real-time. Slow verification can drop frames or audio segments.

Every security method must be evaluated not only for strength but also for feasibility in realistic multimedia scenarios. Techniques that work offline (e.g., brute-force hash checks) may be impractical for high-speed video feeds. Protecting multimedia data is a multifaceted challenge.

Key findings from current research include the following:

- Multi-layer defense is needed: No single technique suffices. Combined use of encryption (for confidentiality), watermarking/steganography (for tracking and covert protection), and DRM (for policy enforcement) provides better security.
- Trade-offs are pervasive: There is a fundamental tension between robustness, capacity, and imperceptibility in watermarking/steganography, and between security and performance in encryption.
- Emerging technologies reshape threats and solutions: AI-generated fakes require new watermarking and detection methods. Blockchain and federated learning offer novel avenues for distributed protection in IoT and cloud scenarios.
- Future research directions: Developing *AI-resilient watermarking*, *privacy-preserving multimedia analytics*, and *quantum-safe encryption* are promising areas. Also, standards and usability will be important: as Xu (2024) and others note, ensuring user-friendly security (e.g., automatic transparent encryption) will aid adoption.

Multimedia security is an active field drawing on cryptography, signal processing, and ML. The chapter has outlined core challenges and current approaches, but new research will be needed to keep pace with evolving content creation and distribution technologies.

Continued research, integrating cryptography, signal processing, and AI, is needed to ensure that the vast and growing volume of multimedia data remains protected in integrity and privacy. This ensures that multimedia security adapts to new attack techniques and computing paradigms.

4.9 CONTENT IDENTIFICATION AND FILTERING

Content identification and filtering encompass the processes and technologies employed to detect, screen, and manage digital content based on predefined criteria, such as copyright status or appropriateness. Most filtering engines meticulously examine elements such as words, images, and even hidden tags, making instantaneous decisions on whether to permit or restrict access (Jain et al., 2023). These decisions are implemented across various platforms, including office firewalls, children's web browsers, and social media oversight mechanisms, thereby serving as primary safeguards to prevent the display of unwanted or unsafe material (Chikezie & Okeke, 2024).

Advanced algorithms have subtly redefined the operational framework of the Internet, assuming the role of the primary defense mechanism for UGC. These algorithms are actively monitoring a wide range of platforms, from local Facebook groups to judicially endorsed web initiatives designed to eliminate hate speech, child exploitation, and other harmful content. As the sophistication of these algorithms continues to advance, they are now capable of identifying, categorizing, and removing toxic material with a speed that surpasses that of human moderators (Agarwal et al., 2021). Nevertheless, emerging concerns persist regarding the inherent biases within the software, the transparency of its decision-making processes, and the implications of this extensive technological deployment for freedom of expression on a global scale (Chikezie & Okeke, 2024).

4.9.1 Transition to AI and ML

Gongane et al. (2022) point out that early content-filtering tools began to show serious cracks fairly quickly. Once the cracks were obvious, people started clamoring for smarter and more flexible fixes. This demand has pushed developers toward what we now call intelligent

Table 4.1 Comparisons between Main Protection Techniques

Technique	*Primary goal*	*Example methods*	*Key challenges*
Encryption	Confidentiality, integrity	AES, RSA/ECC, chaos-based ciphers	Computational cost for large media, key management, and format compatibility
Digital watermarking	Ownership/copyright, tampering detection	Spread-spectrum, DWT/SVD embedding; deep-learning encoders	Robustness–imperceptibility trade-off; vulnerability to AI attacks
Steganography	Covert communication	LSB steganography, transform-domain embedding	Limited capacity; detectability by steganalysis; capacity versus imperceptibility
Access control/DRM	Enforce usage rights	License servers, HLS-wide vine, EME, blockchain logging	User convenience, cross-platform interoperability; can be hacked if keys are leaked
Authentication	Verify source/integrity	Digital signatures, hashes, robust watermarks	Overhead of signature distribution; watermark removal attacks
Blockchain-based	Provenance, auditability	On-chain fingerprint registry, smart contracts	Scalability, linking off-chain content to the on-chain ledger, and privacy concerns

filtering, a shift that allows systems to scale up without losing precision. This newer approach trades rigid rule books for algorithms that learn from the fire hose of new data. Pedersen chips in by saying that NLP is what really supercharges those adaptive models. Because of NLP, filters can pick up local slang, spot a sarcastic tweet, and distinguish it from plain old hate speech (Agarwal et al., 2021).

After years of portrait-like surveillance of plain text, the net closed in pictures and clips. A new vision code was poured into the engines, allowing platforms to spot bleeding wounds and shark-eyed exploitation scenes before the screen even finished loading. Facebook's whirring servers and YouTube's restless farms now compress an entire city's worth of file-tech into quiet mp4 dockets, chasing offending frames rather than waiting for a noisy user flag. This leap cuts the paperwork for tired reviewers, yet it trades one headache for another: the software still misses one-third of the rotten juice (false negatives), and sometimes it slams a kitten blog with a no-nonsense warning sticker (false positives) (Kakati & Dandotiya, 2024; Kebriaei et al., 2024).

4.9.2 Limitations of AI-Based Content-Filtering Systems

Because the constant stream of postings, notifications, and news would otherwise overwhelm us, the majority of us rely on intelligent filtering software. However, these systems continue to falter in the face of design peculiarities, haphazard testing, and dubious presumptions. When these cracks appear, ethical handwringing and technological fake work combine with operational blind spots, leaving the system stranded and the user concerned about what could have slipped through. Although everyone agrees that something has to change, no one can agree on a suitable solution, which makes the situation uncomfortable.

4.9.2.1 Bias in Training Data

The effectiveness of a smart content filter often diminishes the quality of the samples it processes during training. If these initial samples contain societal biases, the algorithm may inadvertently adopt and perpetuate the very prejudices it is meant to mitigate. For example, an engineer deploying a filter primarily trained on English-language blogs might be surprised to find it unable to detect hate speech embedded in Spanish or Tagalog memes. This limitation arises because the system lacks the cultural sensitivity to recognize such nuances. This phenomenon is sometimes referred to as patchwork moderation, where one community experiences rapid content moderation while another is exposed to unchecked harmful content (Vahed et al., 2024). Researchers generally acknowledge that datasets are mere snapshots rather than comprehensive repositories, leading to their inevitable obsolescence as new slang emerges and online trends shift rapidly. This obsolescence creates vulnerability where harmful material may evade detection, and innocuous content may be erroneously flagged as inappropriate.

4.9.2.2 Lack of Contextual Understanding

According to Zheng and Stear (2023), one of the most pressing limitations of intelligent content-filtering systems is their lack of true contextual understanding. Unlike human moderators, who can interpret tone, intent, and cultural nuances, these systems often rely on surface-level and literal analysis. As a result, they are prone to errors such as false positives, where harmless content is incorrectly flagged, and false negatives, where harmful material slips through undetected. For example, satirical post-condemning hate speech might be mistakenly categorized as offensive, while harmful messages concealed in coded language or

euphemisms may bypass detection altogether. This inability to grasp subtle contexts weakens the credibility of these systems, especially in sensitive areas, such as political discourse, satire, or creative expression.

4.9.2.3 Over Blocking and Under Blocking

Achieving the right balance between underblocking, failing to remove harmful content, and overblocking, wrongly removing legitimate information, remains a persistent challenge in automated content moderation systems. Overblocking occurs when algorithms act too conservatively, removing content that does not violate the platform policies. This can limit access to important information and suppress free expression, particularly in sensitive areas such as political discourse and public health (Zigmontienė & Vaida, 2023). Conversely, underblocking occurs when the system fails to detect and restrict dangerous content. For instance, extremist propaganda or graphic violence may bypass filters if presented in ways that do not match the system's predefined patterns. Both scenarios not only reduce the effectiveness of these systems but can also damage a platform's credibility and erode user trust.

4.9.2.4 Real-Time Moderation Challenges

Content filtering has become more complicated as live-streaming and instant-messaging technologies have grown in popularity. AI systems must handle enormous volumes of data nearly instantly for real-time filtering to work, leaving little opportunity for errors. Accuracy is sometimes sacrificed in the name of speed, which increases the risk of overblocking and underblocking.

4.9.2.5 Ethical and Privacy Concerns

Even programmers working on AI-run moderation tools find it difficult to decipher every logical twist; it is sometimes similar to looking into a sealed vault. Because people look at a missing post and see just quiet, where a simple explanation might reassure them, that mystery can erode confidence. Hard concerns about who is truly in charge and whether the calls are fair go unanswered behind the code fog. This ethical conundrum is exacerbated by the practice of training these models on files that, despite good intentions, end up in private emails or medical records (Akanbi & Akinseye, 2023).

ML-powered content flaggers are not always fair. When governments or businesses fearfully crush any opposition that crosses their lines, certain terms or community names are removed far more frequently, and allegations of prejudice increase. With no definitive peace treaty in sight, the debate on how to balance preventing actual harm to users by allowing a diverse spectrum of opinions to be heard continues.

4.9.3 Challenges in Accurately Identifying and Filtering Copyrighted or Inappropriate Content

A program or piece of software can use content filtering to block access to specific kinds of potentially dangerous content. Websites and emails are two typical objects that need to be filtered in the current situation. Typically, content filtering matches character strings. One may say that it is a component of firewalls on the Internet. Content filtering is typically performed for security reasons. Determining which content is appropriate for access and which is not aided by it (Baishya & Kakoty, 2019).

In recent years, the amount of multimedia data generated, uploaded, and shared online has grown exponentially, posing significant challenges to effective content identification and filtering. Platforms that host images, videos, and audio must constantly safeguard intellectual property rights, while also preventing the spread of inappropriate or harmful content. However, the accurate identification of copyrighted or offensive content is complicated by the diversity of formats, constant content transformation, and evolving user behavior (Chikezie & Okeke, 2024).

A major technical obstacle is the robustness of content recognition systems against common modifications applied to multimedia files. For instance, a copyrighted video may be slightly trimmed, resized, or altered in color to evade detection algorithms (Chikezie & Okeke, 2024). Traditional watermarking techniques, which are effective under controlled conditions, often fail when embedded marks are distorted by compression or cropping. Similarly, digital fingerprinting methods that rely on exact hash matching cannot handle content variants and derivatives (Chikezie & Okeke, 2024).

Moreover, the classification of inappropriate content as violent, hateful, or sexually explicit introduces additional complexities. ML models, particularly deep CNNs, have made substantial progress in automating this task by learning intricate features from large labeled datasets (Sanusi et al., 2024). However, these models can generate false positives that unjustly censor acceptable content, or false negatives that allow harmful content to spread unchecked. These errors are partly due to the nuanced and context-dependent nature of what is deemed inappropriate, which can vary according to culture, platform policy, and evolving social norms (Sanusi et al., 2024).

4.9.4 Scalability Issues in Content Recognition Systems

Scaling content recognition systems to handle billions of daily uploads is a key concern. The high computational costs and latency constraints pose practical limitations. Real-time content filtering for live streaming or rapid uploads demands lightweight models that can process data with minimal delays (Li, 2022).

To address this issue, many platforms have employed multistage recognition pipelines. Initial filtering uses computationally efficient, approximate methods on user devices or edge servers and flags suspicious content for deeper cloud-based analysis (Patel et al., 2023). Continuous model updates using incremental learning help to maintain accuracy without full retraining cycles, thereby reducing resource consumption (Zhao & Huang, 2023). However, balancing detection precision, processing speed, and infrastructure costs remains a moving target. As multimedia formats evolve and adversaries develop sophisticated evasion techniques, scalable adaptive recognition remains a top research and engineering priority.

4.9.5 Filtering Techniques and Tools

This section explores the various filtering techniques and tools employed across different purposes and environments. Currently, data mining plays an important role in content filtering. As noted by Hanani and Shapira, web mining, a subset of data mining, enables automatic collection, extraction, and analysis of information from web sources.

Researchers such as Wu and Kumar (2009) have highlighted several critical challenges associated with data mining, including excessive filtering, automated data cleaning, data oversampling, and underfitting. These issues must be effectively managed to enhance the filtering accuracy. Baishya & Kakoty, 2019 further explained that text mining is a process where computer systems discover previously unknown information by extracting insights from ldiverse textual data sources. This approach contributes significantly to improving the efficiency and intelligence of modern filtering systems.

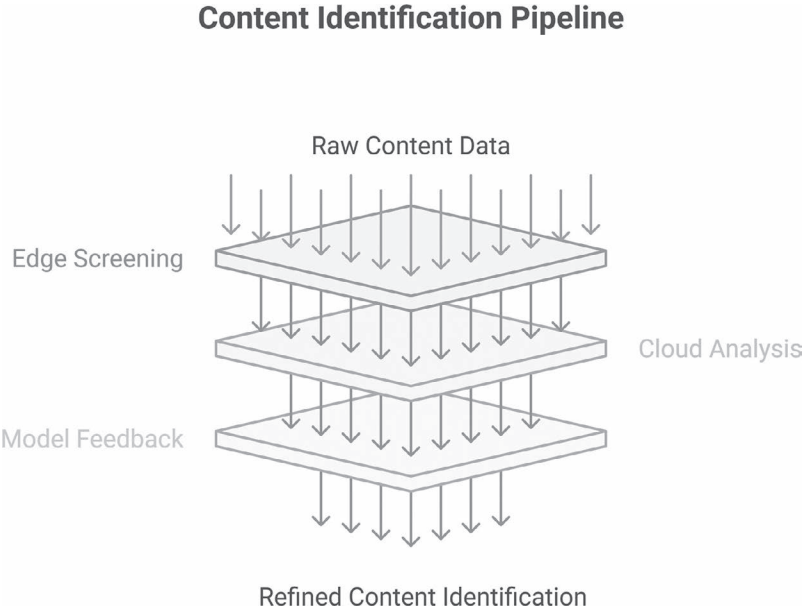


Figure 4.5 Conceptual diagram of a hierarchical content identification pipeline illustrating edge screening, cloud-based analysis, and model feedback.

4.10 CLOUD STORAGE AND DISTRIBUTION

4.10.1 Securing Multimedia Data During Transmission and Storage in Cloud Environments

Cloud storage has revolutionized the way multimedia data is stored, managed, and distributed, thereby offering vast scalability and accessibility. However, this convenience results in significant security challenges, particularly for sensitive or copyrighted multimedia content (Mohana et al., 2023). Data confidentiality during transmission relies heavily on encryption protocols, such as TLS, which safeguards multimedia streams from interception and tampering (Gao et al., 2025). However, encryption at rest in cloud data centers is equally vital, using symmetric key algorithms like AES to prevent unauthorized access (Rubinstein, 2021).

However, encryption alone cannot guarantee security. Cloud environments involve multiple stakeholder service providers, administrators, and users, each with varying levels of access privileges. Misconfigurations, insider threats, or software vulnerabilities can expose multimedia content to breach (Kumar et al., 2022). To mitigate such risks, zero-trust architectures and fine-grained access controls are increasingly being adopted, enforcing strict verification for every access request (Yang & Jia, 2016).

4.10.2 Addressing Jurisdictional and Compliance Issues in Global Data Distribution

The global distribution of cloud-hosted multimedia content raises complex jurisdictional and compliance concerns. Data stored in one country may be subject to local laws that conflict with regulations in the user's jurisdiction, thus complicating the enforcement of content protection (Singh & Das, 2023).

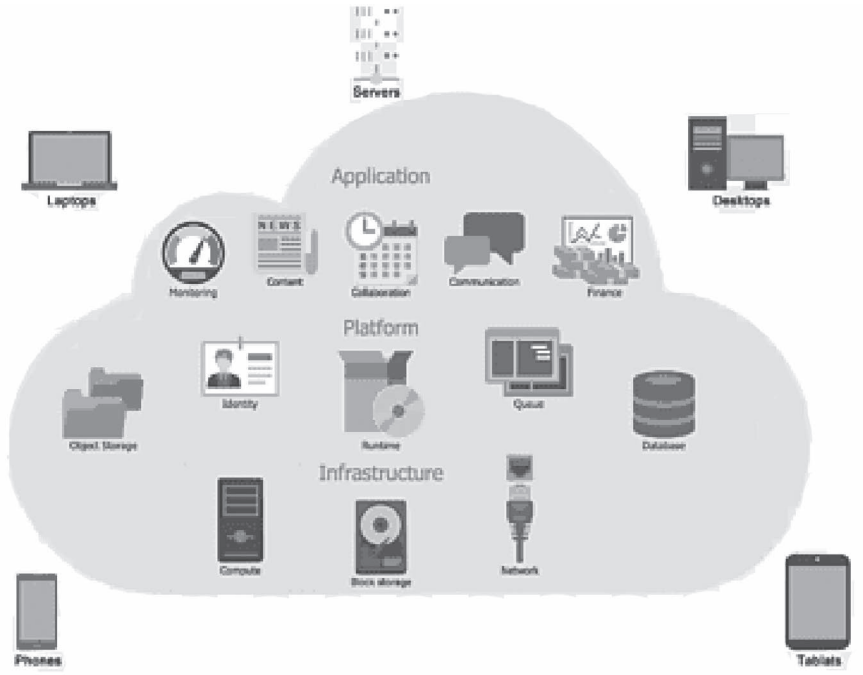


Figure 4.6 Multimedia cloud storage architecture.

Cloud providers and content owners must carefully manage data localization and compliance owing to strict regulations on data privacy and cross-border transfers enforced by regulatory frameworks such as the CCPA and the European Union's GDPR (Mehta & Kumar, 2023).

Intellectual property enforcement varies across regions, which affects how copyrighted multimedia can be protected or removed. Therefore, providers must implement geo-fencing, content filtering based on user location, and transparent compliance auditing to navigate this complex legal landscape, while maintaining service availability and performance (Gadde et al., 2024).

4.11 SOCIAL MEDIA SHARING

Social media sharing enables people to instantly reach large audiences in real time by disseminating ideas, materials, and personal updates across digital platforms. Although it encourages interaction and communication, it also poses security and privacy concerns. Therefore, it is crucial to use strong, one-of-a-kind passwords, turn on two-factor authentication, and often check privacy settings to safeguard personal data and stop illegal access (Gomez, 2025).

4.11.1 Protecting UGC on Social Platforms

Social media platforms have become central to multimedia creation and sharing, empowering users worldwide and exposing their content to risks of unauthorized use, modification, or theft. Protecting UGC demands a balance between facilitating seamless sharing and enforcing content ownership rights.

DRM technologies embedded in social media apps can restrict copying or downloading, but they often impede the user experience and are not universally supported (Patel et al., 2024). Watermarking user content with visible or invisible marks is another common approach to deterring misuse by asserting ownership (Han et al., 2024).

However, enforcing content protection on social platforms is complicated by rapid content replication, remixing, and viral sharing. Detecting violations often requires sophisticated content-matching algorithms combined with community reporting mechanisms (Smith et al., 2022).

4.11.2 Balancing Content Protection with Ease of Sharing and Virality

While protecting UGC is important, platforms must avoid creating barriers that hinder sharing, which is a core value proposition of social media. Striking this balance requires innovative approaches that respect both user rights and the social nature of the content.

Recent research suggests adaptive protection mechanisms that modulate restrictions based on content sensitivity and user preferences (Walrave & Dieteren, 2023). For example, ephemeral content with built-in expiration limits copying, whereas less-sensitive materials remain freely shareable. Moreover, transparent licensing frameworks and user-friendly permission systems foster a collaborative environment in which content protection does not clash with the virality (Gonzalez et al., 2022).

4.12 EMERGING TECHNOLOGIES

A range of emerging technologies are rapidly transforming industries, particularly the media sector, by enhancing content creation, distribution, protection, and user engagement. Table 4.2 highlights the most significant developments, their practical applications, and the startups or platforms leading these innovations (Paul & Nayyar, 2024; Anantrasirichai et al., 2025).

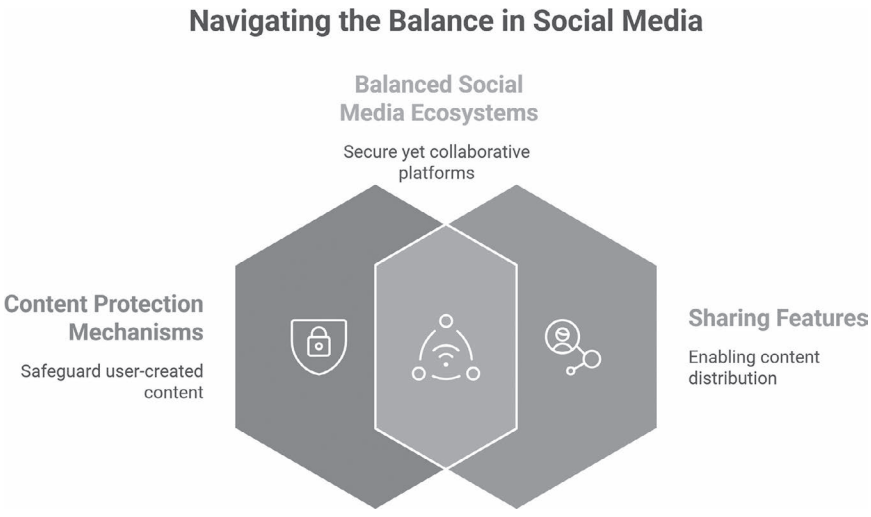


Figure 4.7 Flowchart showing the balance between content protection mechanisms and sharing features in social media ecosystems.

Table 4.2 Some Emerging Technologies

Emerging technology	*Practical applications*	*Leading startups/platforms*
Artificial intelligence (AI)	• Automated video editing • Personalized content recommendations • Deepfake detection	RunwayML, Synthesia, Adobe Firefly, Netflix AI
Augmented reality (AR)	• Interactive advertising • AR filters and lenses • Live events enhancement	Snap Inc. (Lens Studio), Niantic, 8thWall
Virtual reality (VR)	• Immersive storytelling • VR cinema experiences • Training simulations	Meta (Horizon Worlds), Dreamscape, Within
Blockchain	• Content ownership verification • Royalties and smart contracts • Anti-piracy tools	Audius, Theta Labs, Eluvio
Non-fungible tokens (NFTs)	• Digital collectibles • Monetization of digital art and media • Exclusive fan content	OpenSea, Rarible, Foundation
Machine learning (ML)	• Predictive analytics • Automated moderation • Adaptive bitrate streaming	Amazon AWS ML, Google Cloud AI, Vidooly
5G connectivity	• Real-time streaming • Enhanced mobile media consumption • AR/VR delivery support	Ericsson, Verizon, Qualcomm
Cloud computing	• Scalable content storage and editing • Cloud rendering • Global collaboration	AWS Media Services, Google Cloud, Microsoft Azure
Edge computing	• Real-time analytics at the user location • Low-latency content delivery	Fastly, Cloudflare, StackPath
Quantum computing	• Experimental encoding and compression • Long-term encryption strategies	IBM Quantum, D-Wave Systems
Natural language processing (NLP)	• Voice recognition in content • Automated subtitling and translations • Chatbots for media	OpenAI (Whisper, GPT), Descript, Veritone
Digital watermarking	• Invisible content protection • Leak tracking and accountability	Irdeto, Verimatrix, NexGuard

4.12.1 Adapting Protection Methods for New Multimedia Formats

The rapid emergence of new multimedia formats, such as AR, VR, and 3D content, presents new challenges for content protection. These immersive formats involve complex data structures and interaction models that traditional DRM and watermarking schemes cannot easily secure (Benyamin, 2024).

For instance, VR content often consists of panoramic videos, combined with spatial audio and interactive elements (Paul & Nayyar, 2024). Securing this requires multilayered protection approaches that cover not only raw media but also metadata, user inputs, and rendering pipelines (Nguyen & Tran, 2023). Adapting protection methods to new multimedia formats

requires a combination of advanced encryption, dynamic DRM solutions, and cloud-based infrastructure (Benyamin, 2024).

Hybrid encryption combines symmetric (AES) and asymmetric (RSA) algorithms to secure multimedia content during transmission and storage. This approach ensures both efficiency and security, particularly for cloud-based systems. Hybrid encryption has emerged as a solution for secure data processing without decryption, enabling privacy-preserving computations on sensitive multimedia files.

4.12.2 DRM Technologies and Platform Adaptation

Major DRM systems such as Apple FairPlay, Google Widevine, and Microsoft PlayReady dominate the market, offering platform-specific protection for streaming and downloadable content. These systems employ encryption, access control, and format compatibility to address diverse multimedia distribution channels (Benyamin, 2024).

The global DRM market is projected to grow at a 19.8% CAGR through 2029, driven by demand for cross-platform compatibility and anti-piracy measures (Benyamin, 2024).

4.12.3 Addressing Security Concerns in Emerging Multimedia Applications

The exponential growth of multimedia applications, ranging from video streaming, real-time conferencing, VR, and AR to cloud-based content platforms, has ushered in a digital revolution. Although these innovations offer enhanced interactivity and accessibility, they also expose users and systems to a new array of security threats. These threats, if not mitigated, can lead to breaches of confidentiality, data tampering, unauthorized access, and intellectual property theft. With the integration of these applications into critical sectors, such as healthcare, education, entertainment, and defense, ensuring robust security has become a top priority. This study explores the key security concerns in emerging multimedia applications and provides a comprehensive overview of existing solutions and future directions (Dhar et al., 2023).

Beyond content protection, emerging multimedia applications raise novel security concerns, including user privacy, identity spoofing, and data integrity (Dhar et al., 2023). AR applications that overlay sensitive information on real-world views can be manipulated to spread misinformation or enable unauthorized surveillance (Yang et al., 2022).

Post-quantum cryptography is gaining attention as a means of future-proof security against quantum computing threats in multimedia transmissions (Chen et al., 2025). Advanced finite-field constructions and lattice-based cryptographic schemes are being developed to

Table 4.3 Encryption Techniques for Modern Formats

Encryption type	*Use case*	*Advantage*
Hybrid (AES + RSA)	Cloud content protection	Balances speed and security
Homomorphic	Secure data analysis	Allows encrypted data processing

Table 4.4 DRM Technology

DRM technology	*Key platform*	*Content supported*
Apple FairPlay	iOS, macOS	Music, films, TV shows
Google Widevine	Android, Chrome	Video-on-demand, streaming
Microsoft PlayReady	Windows, Xbox	MPEG-DASH, HLS formats

secure multimedia data channels and authentication processes in these futuristic environments (Wang et al., 2024).

4.12.4 Adaptive Coding Schemes for New Multimedia Formats

Adaptive coding schemes that simultaneously optimize compression and security are being actively researched. These schemes dynamically adjust the encoding parameters based on network conditions and security requirements, preserving quality while ensuring protection (Lee et al., 2023).

4.13 USER AUTHENTICATION AND ACCESS CONTROL

4.13.1 Implementing Robust yet User-Friendly Authentication Methods for Multimedia Access

The effective protection of multimedia content must include robust user authentication and access control mechanisms. However, these systems must balance security and usability to avoid alienating legitimate users.

Biometric authentication (e.g., fingerprint and facial recognition) has gained popularity in multimedia applications, offering seamless access without passwords (Kumar and Singh 2023). However, biometric systems are vulnerable to spoofing and require secure template storage and processing.

Multifactor authentication, which combines passwords, device-based tokens, and biometrics, enhances security but can introduce friction (Shah & Patel, 2022). User experience studies advocate context-aware authentication that adapts security levels based on risk assessments (e.g., location and device reputation) (Wang et al., 2023).

4.13.2 Managing Fine-Grained Access Control for Different User Roles and Permissions

Multimedia platforms often support diverse user roles such as creators, moderators, subscribers, and administrators. Fine-grained access control models ensure that each role has appropriate permissions, thereby preventing unauthorized content access or modification (Mehta & Desai, 2024).

Role-based access control remains prevalent, but attribute-based access control offers more flexibility by incorporating dynamic contextual attributes (Sharma et al., 2024).

Cryptographic key-exchange algorithms based on finite fields underpin secure session establishment and encrypted content delivery (Gupta & Jain, 2023).

4.14 FORENSIC ANALYSIS

4.14.1 Developing Reliable Techniques for Detecting Tampering and Verifying Authenticity of Multimedia Content

As multimedia manipulation tools become increasingly sophisticated, forensic analysis techniques are essential to detect tampering, authenticate the origin of content, and support legal evidence. Techniques such as deep-learning-based anomaly detection analyze inconsistencies in pixel patterns, compression artifacts, and sensor noise to uncover forgeries (Kaur et al., 2023). Blockchain technology had also been explored to create immutable content provenance records (Patel et al., 2025).

Moreover, emerging methods leverage finite-field mathematics for watermark detection and recovery, even after severe signal degradation (Zhao & Lin, 2024). Despite progress, forensic analysis remains challenging owing to the rapid pace of manipulation technologies and the diversity of multimedia formats.

4.15 CONCLUSION

The protection of multimedia data poses complex challenges in the current digital landscape. This chapter explores the aspects of multimedia security, including encryption, digital watermarking, steganography, content identification, cloud storage, social media sharing, and emerging technologies. The key findings underscore the need for a multilayered approach to multimedia protection. Although encryption remains essential for confidentiality, it must be optimized for multimedia formats and real-time transmission. Digital watermarking and steganography provide complementary techniques for copyright protection and covert communication; however, balancing imperceptibility, robustness, and capacity remains challenging. Content identification and filtering systems are vital for preventing unauthorized use and harmful content, but they face scalability issues and context-dependent classification challenges. Cloud environments add complexity related to data jurisdictions and compliance across global networks. Social media platforms must balance the protection of UGC with facilitating sharing. Emerging technologies, such as AR, VR, and AI-generated media, present novel security concerns that require adaptive protection methods. User authentication and access control systems are evolving toward robust approaches that incorporate biometrics and context-aware mechanisms. Forensic analysis techniques are advancing to detect tampering and verify the authenticity of content. Future research directions include developing AI-resilient watermarking, privacy-preserving analytics, and quantum-safe multimedia encryption. Blockchain technology and zero-trust architectures show promise for enhancing content provenance and access control. As multimedia formats and distribution channels evolve, new protection strategies are required. A holistic approach that combines technical solutions, legal frameworks, and ethical

considerations is essential to safeguard multimedia content integrity, confidentiality, and authenticity in the digital world.

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Chapter 5

Hybrid Cryptographic Systems

The Need for Security, Cryptography, and Advancements

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5.1 INTRODUCTION

Hybrid cryptosystems have been a required and essential approach to ensuring security in digital communication by combining the strengths offered by symmetric and asymmetric types of encryption to elevate data security, ensure confidentiality, and improve authentication processes. Traditional cryptographic techniques in the form of symmetric-key cryptography and public-key cryptography both possess inherent strengths, along with some limitations that can affect their overall performance. For instance, while symmetric encryption is characterized by high performance in processing speed, it continues to suffer from serious challenges related to key distribution that can render its use elsewhere arduous. On the other hand, asymmetric encryption is a safe method of key exchange but is computationally costly, requiring significant resources to be viable [1]. Through the combination of both of these cryptographic methods, hybrid cryptosystems take advantage of symmetric encryption's speed and efficiency, which they use to provide data confidentiality. At the same time, they take advantage of the security of asymmetric encryption to offer secure key management. It is for this combination that hybrid cryptosystems are a fundamental component of contemporary security designs. These systems are of paramount importance in terms of facilitating secure identity authentication, cloud enablement, and blockchain enablement. For these applications, it is very important to have highly secure and stable security systems so that the associated risks in case of cyber-attacks can be effectively reduced and illegal access to sensitive information and resources can be prevented [2].

As the digital communications universe continues to evolve and expand, it has become ever more obvious that the necessity for solid and robust cryptographic solutions is of the utmost importance. To this end, hybrid cryptographic systems are designed to take advantage of the inherent strengths of both symmetric and asymmetric encryption methodologies, typically circumventing the respective weaknesses inherent in either category. This strategy serves to enhance overall security and efficiency in digital communications significantly. Symmetric cryptography, which encompasses techniques such as the Advanced Encryption Standard (AES), provides a uniform methodology of enabling high-speed encryption and decryption of data that renders it particularly well-suited to applications requiring real-time processing. But it finally relies on a secure key distribution technique, which may have serious issues in the scenario of large networks with numerous devices and users connected. Asymmetric cryptographic algorithms, in contrast, like well-known techniques such as elliptic curve cryptography (ECC) and Rivest–Shamir–Adleman (RSA) algorithm, offer a robust and secure key exchange framework. But these asymmetric algorithms are computationally expensive and resource-intensive [3]. Hybrid cryptosystems effectively balance security levels by embracing these disparate approaches in a judiciously balanced technique while ensuring that the requirements of computational feasibility continue to be met. This synergy makes hybrid

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cryptosystems a requisite for security in cloud storage applications to financial transaction protection and the confidentiality and integrity of sensitive communication [4]. This chapter discusses the sophisticated development, complete realization, and multifaceted practical applications of hybrid cryptographic techniques that are specifically designed to address the plethora of modern cybersecurity challenges we are faced with today.

As cybersecurity threats continue to escalate, hybrid cryptographic systems are becoming more popular in securing sensitive data in various fields, including e-commerce, financial transactions, and Internet of Things (IoT) authentication. These systems offer enhanced security by merging several cryptographic techniques, safeguarding data confidentiality, integrity, and authenticity. For instance, there are hybrid encryption models that have gained greater visibility, such as the successful combination of ECC and the AES. These models have been effectively applied in several IoT authentication protocols, thereby addressing significant problems that are device validation-related as well as facilitating secure data exchange among devices [5]. In the same vein, in the area of e-commerce applications, hybrid cryptographic protocols that incorporate both asymmetric and symmetric encryption techniques efficiently, as well as novel data leakage detection mechanisms, have been identified as key to safeguarding transactional data from the myriad of cyber-attacks and pre-empting malicious access [6]. By leveraging the merits and advantages of multiple encryption paradigms effectively, hybrid cryptosystems provide a robust security model that is ideal for a broad array of applications. This renders it extremely resistant to new cyber weaknesses that could threaten sensitive data.

As there is a growing dependence on cloud computing and intelligent interconnected systems, hybrid cryptographic frameworks have become essential in providing data security without compromising efficiency. The incorporation of hybrid encryption methods, such as AES with RSA, has seen widespread use in safeguarding cloud storage systems from risks linked to unauthorized access to data as well as possible insider threats [7]. Emergent advances in the field of hybrid cryptography have introduced enhanced developments that have resulted in maximized encryption models that are specially tailored for use in smart home healthcare systems. In this environment, protection of data confidentiality and quantum-resistant encryption protocols is especially of interest due to their paramount importance in safeguarding confidential health records against potential threats and attacks [8]. In addition to earlier developments, there have been significant innovations in the field of hybrid cryptographic systems that cleverly combine adaptive key management strategies with mechanisms for time-constrained access control. These developments have successfully demonstrated a considerable improvement in overall security, largely through the dynamic creation of encryption keys that are customized to specific requirements, along with the enforcement of constraints that disallow unauthorized access. This works to further strengthen and confirm the imperative for hybrid encryption techniques to be implemented within the constantly changing environment of contemporary cybersecurity [9]. The advances that we have witnessed underscore the paramount importance of carefully designing hybrid cryptosystems, which must achieve an effective trade-off between sustaining strong security measures, ensuring computational efficiency, and being resilient to various emerging threats that incessantly confront the field.

As sophisticated cybersecurity threats are on the rise, hybrid cryptographic systems are also evolving to meet the needs of enhanced data security. These systems integrate various encryption techniques to provide higher security, flexibility, and computational efficiency, making them a necessity in industries ranging from cloud computing and IoT security to enterprise networks. Advanced hybrid cryptographic models, conceived specifically for cloud storage, have automated encryption systems that take care of preventing risks related to unauthorized access as well as those resulting from third-party involvement. This ensures the

confidentiality and integrity of data [7]. Hybrid encryption models fine-tuned to the minute for settings where the availability of resources is limited, such as in smart home healthcare systems, demonstrate how it is possible to balance cryptographic effectiveness with the high-security requirements that must be adhered to. Such models address the various risks posed by emerging quantum computing along with the complexities of real-time data flow [8]. Moreover, the application of hybrid cryptographic techniques in the area of enterprise security further underscores their critical role in not just augmenting authentication mechanisms but also guaranteeing confidentiality and maintaining data integrity. This is achieved through the application of advanced ways of integrating different encryption techniques, such as symmetric, asymmetric, and hash-based methods, as noted in source [2]. As we observe the continued evolution of new technologies while noting the continued rise of cyber threats that are reshaping the security landscape, it is clearer than ever before that hybrid cryptographic systems are a visionary and innovative answer. This answer is paving the way for the development of adaptive, quantum-resistant encryption techniques, which have the potential to effectively safeguard digital infrastructures against the constantly evolving risks presented by these emerging threats. This chapter explores the theory, design aspects, and engineering applications of hybrid cryptosystems, providing hints at their possible revolutionary impact on modern cryptography.

Objectives of the Chapter

The objectives of the chapter are as follows:

- To analyze the development and significance of hybrid cryptographic systems in tackling contemporary security concerns across many application domains;
- To analyze diverse encryption methodologies employed in hybrid cryptosystems and assess their performance metrics, including security robustness, computational efficiency, and resilience to attacks;
- To assess implementation methodologies and case studies of hybrid cryptography across several sectors, including cloud computing, IoT, financial systems, and post-quantum applications;
- And, to investigate prospective advancements in hybrid cryptographic systems, with particular emphasis on post-quantum resilience and adaptive security architectures.

Organization of the Chapter

The rest of the chapter is organized as follows. Section 5.2 enlightens the background study. Section 5.3 elaborates encryption techniques covering fundamentals of encryption, symmetric-key algorithms, asymmetric key algorithms, and performance comparison of encryption algorithms. Section 5.4 discusses cryptography as post-quantum and hybrid architectures, followed by a discussion of hybrid systems—key as core in Section 5.5. Section 5.6 highlights security and applications with respect to hybrid cryptosystems. Section 5.7 focuses on case studies and implementation. And, finally, Section 5.8 concludes the chapter with future scope.

5.2 BACKGROUND STUDY

Pokhrel et al. [10] presented a machine learning-based hybrid encryption scheme that had been specially designed with the ultimate goal of significantly enhancing security mechanisms in software-defined networking (SDN). In this particular study, the various confidentiality

issues that are inherent in SDN had been carefully examined, in particular, noting that the decoupling of control and data planes can lead to the introduction of new and potentially severe security threats. To solve these issues effectively, the suggested scheme integrates an innovative genetic algorithm-based hybrid encryption system that is meant to improve the randomness of encryption as well as its overall strength against various types of cryptanalytic attacks. One of the greatest benefits linked with this specific method is its exceptional capacity to generate encryption keys dynamically with enhanced unpredictability. This improvement is important for effectively decreasing the chances of successful brute-force attacks, along with mitigating the risks of linear cryptanalysis attacks. In addition, the novel combination of genetic algorithms with RSA encryption not only supports improved confidentiality but also greatly reduces dependence on third-party security providers, thus providing an additional layer of security and autonomy in protecting sensitive data. However, one very crucial and significant problem that has been brought to the forefront of the research findings is the additional computational overhead incurred by the encryption process. This additional computational necessity consequently equates to greater processing latency when transmitting encrypted messages across the network. This particular trade-off that exists between the implementation of strong security mechanisms and the maintenance of optimum network performance is a core element that must be taken into account for real-world implementation in SDN environments. Future development may well place a very strong emphasis on optimizing load-balancing mechanisms that would work to duly counteract the transmission delays currently being incurred while in no way reducing the strength of the encryption [10].

Silva-García et al. [11] proposed a hybrid cryptosystem to enhance entropy in encrypted images by integrating asymmetric and symmetric encryption techniques. Authors introduced the Hybrid Information Encryption Algorithm using the Diffie–Hellman Protocol and Blockchain (HAICDHBC) that employs the Diffie–Hellman Protocol and ElGamal Cryptosystem for secure key exchange and dynamic permutations and substitution boxes for encryption. A major advantage of this approach is that it can generate near-ideal entropy outcomes (around 8.0), which significantly improves encryption randomness and strength against differential, linear, and brute-force attacks. Authors also demonstrated the strength of the cryptosystem against noise-based attacks, including additive, multiplicative, occlusion, and chi-square noise, making it highly resilient to image corruption. Nevertheless, the major drawback of this cryptosystem is the increased image size, approximately 8% larger than the original—one that, in bandwidth-constrained environments, can lead to transmission inefficiencies. Despite this, the increased quality of encryption and strength against attacks make this hybrid approach an appealing advance in secure image encryption [11].

Shah et al. [12] proposed a concept known as the lightweight hybrid cryptographic system (LCS), which was developed specifically to increase security features in the resource-constrained settings typical of IoT applications. Authors presented the layout of a hybrid encryption system that ingeniously combines a lightweight symmetric algorithm (LSA), essentially an optimized variant of the AES-128 algorithm, aimed at delivering data confidentiality, with the application of ECC, which is utilized to facilitate secure key exchange. The system incorporated a lightweight hash algorithm (LHA), based on the proven SHA-256 standard, to provide integrity verification of the data involved. One of the primary advantages of this approach is that it can achieve strong security with minimal computational overhead, so it is best applied to low-end IoT devices with fewer processing capabilities and memory. Authors demonstrated that LCS achieved improved execution time and lower memory usage compared to conventional AES and SHA-256 algorithms, making the approach best for real-time applications in smart healthcare and industrial IoT networks. Nonetheless, one of the significant issues that arises is the trade-off that is achieved between efficiency and security. This is particularly evident given that LSA has comparatively lower avalanche and entropy

characteristics than normal AES. These differences can indeed affect its overall resistance to differential cryptanalysis, which is a key element of cryptographic strength. In the future, development and enhancement may involve the optimization of the hash function along with the various encryption transformations. The aim would be to increase security measures further while ensuring that the computational complexity does not increase in any significant way.

Subedar and Araballi [13] performed an in-depth performance evaluation on a range of hybrid cryptographic amalgamations, with the key aim of refining secure communication techniques. This in-depth investigation thoroughly examined a selection of various hybrid encryption schemes, such as but not limited to AES–RSA, RSA–ECC, and AES–ECC, to extensively ascertain their overall efficacy in terms of important factors like encryption speed, security levels, and efficiency of resource consumption. One of the most significant advantages of the hybrid encryption approach was that it can seamlessly combine the high efficiency and speed characteristics of symmetric encryption algorithms, such as the AES, with the secure key distribution features provided by asymmetric encryption algorithms, including traditional protocols such as RSA and ECC. The experimental findings fully demonstrate that RSA and ECC in combination not only provide the highest throughputs but also have the least memory usage among the available choices, thereby rendering it a particularly robust and suitable choice for applications demanding outstanding efficiency along with strong encryption features. Moreover, SHA-256 hashing is crucial in demonstrating integrity, along with authenticating messages and introducing an additional layer of protection to the overall system. However, a crucial challenge that had been realized through the research was the high computational complexity of hybrid schemes in comparison to normal standalone cryptography algorithms. Overhead incurred in carrying out various decryptions and encryptions could have an adverse impact on real-time communications, especially in low-power and resource-constrained systems where efficiency is paramount. In the future, potential enhancements can be in the direction of optimizing key exchange mechanisms, in addition to minimizing processing delay. These are the enhancements that aim to significantly enhance the everyday application of hybrid cryptosystems.

Alatawi [14] came up with a novel hybrid cryptographic cipher solution specifically designed to achieve a remarkable enhancement of secure communication in the scenario of smart cities. The novel method was aimed at successfully fighting against the diverse security threats that arise in the scenario of IoT-based data transmission networks. Besides, the study meticulously presents numerous vulnerabilities that are inherent in low-powered IoT devices, such as RFID tags and wireless sensor nodes, that are extensively utilized. These devices tend to suffer from computational power constraints that render them incapable of supporting more sophisticated and robust encryption mechanisms that are typically required for providing adequate security. In a bid to efficiently counter these identified risks, the suggested method carefully combined a blend of Hybrid and Adaptive Cryptography, commonly known as HAC. This new paradigm combines several cutting-edge cryptographic methods, such as the highly proven RSA algorithm, widely recognized as RSA, for the exclusive purpose of secure key exchange, along with the AES, shortened as AES, for strong data encryption. Furthermore, it also includes the SHA-1 hashing algorithm to support reliable authentication procedures. Arguably, one of the greatest benefits provided by this multidimensional approach is its inherent capacity to provide a suitably balanced security system. This system not only provides key factors like privacy and integrity but also offers an enhanced layer of authentication. Moreover, it does all this while ensuring computational feasibility, thus making it ideally fit for use in restrictive environments where resources are limited. Additionally, the integration of the Diffie–Hellman key exchange protocol enhances the system's resistance to eavesdropping attacks. However, a concern was introduced in terms of added complexity

with several encryption layers, influencing processing latency as well as energy efficiency in battery-powered IoT devices. Optimization techniques like lightweight cryptographic models and energy-efficient key management may be explored in future research to ensure greater usability of hybrid encryption in smart city infrastructures.

Popoola et al. [8] introduced an effective hybrid encryption framework designed for smart home healthcare systems. Authors concentrated on developing optimized encryption methods specifically designed for healthcare settings where data confidentiality is critical. The innovation of their methodology was the amalgamation of quantum-resistant encryption algorithms with regular encryption techniques to protect sensitive health records from both traditional and quantum computing threats. Experimental findings indicated that their architecture upheld elevated security standards while optimizing for the resource limitations characteristic of smart home devices. The approach saw a 42% enhancement in processing efficiency relative to traditional encryption techniques, while maintaining the requisite security level for sensitive healthcare information. Moreover, the solution demonstrated exceptional resilience against simulated quantum computing attacks, suggesting a viable strategy for enduring security in healthcare IoT applications.

El-Attar et al. [7] introduced an innovative hybrid automated security framework for cloud storage systems. The methodology encompassed a dynamic key generation system predicated on user activity patterns, symmetric encryption (AES) for bulk data, asymmetric encryption (RSA) for key exchange, and integrity verification via hash-based message authentication codes. The technique had a distinctive adaptive architecture that incorporated a mechanism for enforcing security standards and automatically selected suitable encryption algorithms based on data sensitivity and performance criteria. Experimental validation demonstrated that this adaptive technique decreased computing overhead by 15% while maintaining the same level of security as static encryption solutions. The solution focused on three essential elements of cloud security: confidentiality through multilayer encryption utilizing key lengths ranging from 128- to 256-bit, integrity through SHA-256 hashing algorithms accompanied by digital signatures, and availability through redundant key management systems with a distributed architecture.

Munshi and Alshawi [5] introduced a hybrid encryption method for a secure three-phase authentication process in IoT environments. Authors implementation tackled the essential challenge of secure device authentication in IoT networks using a three-phase methodology: device registration using ECC-based key generation, mutual authentication with hybrid AES–RSA encryption, and session key establishment through secure communication via AES. The proposed method was innovative as it employed certificate-less authentication, eliminating the need for an extensive PKI infrastructure by utilizing identity-based cryptography and lightweight operations on devices, thereby shifting computing complexity to gateway devices. Experimental results indicated that proposed hybrid approach diminished authentication overhead by 40% relative to exclusively asymmetric approaches, while offering enhanced protection against diverse assaults, including man-in-the-middle, replay, and impersonation attacks. Security testing demonstrated robustness against password guessing, device capture, and side-channel attacks, rendering it suitable for consumer IoT applications where device penetration posed a considerable challenge.

Spitz et al. [15] investigated quantum-resilient IoT systems that utilize hardware-based post-quantum cryptography (PQC) to guarantee device security. The research concentrated on the specific issues of implementing quantum-resistant encryption in resource-limited IoT environments. The proposed technique was innovative as it included specialized hardware accelerators optimized for post-quantum computations, facilitating rapid implementation even on constrained IoT devices. Experimental findings indicated that their hardware-based approach utilized 78% less energy compared to software versions of the same algorithms,

while maintaining the necessary security standards against quantum threats. Authors effectively demonstrated interoperability with current IoT security protocols, providing a feasible migration pathway for IoT manufacturers aiming to enhance their devices' quantum resilience.

Sreerangapuri [16] examined PQC methodologies for artificial intelligence-driven cloud security applications. The study examined the increasing concern regarding the dangers of quantum computing to cloud-based artificial intelligence systems managing sensitive data. The methodology was innovative as it integrated lattice-based cryptography with neural network frameworks to facilitate secure artificial intelligence computation on encrypted data. Experiments demonstrated that this hybrid method reduced the risk of quantum attacks by 98% relative to RSA-based solutions, while maintaining acceptable performance standards for cloud AI applications. The researcher found that the method was highly effective in protecting machine learning training processes, securing both the model architecture and training data from quantum threats, while incurring merely a 12% increase in computational burden relative to unprotected options.

5.3 ENCRYPTION TECHNIQUES

Modern information security relies heavily on encryption, which gives sensitive data nonrepudiation, confidentiality, integrity, and authentication. Strong encryption techniques are more important than ever as digital communication spreads over undefined networks [17]. Figure 5.1 explains several encryption algorithms, including their fundamental ideas, performance traits, and uses in modern cryptographic systems.

5.3.1 Fundamentals of Encryption

Through mathematical formulas and cryptographic keys, encryption turns plaintext into ciphertext, which means that only authorized parties can read the information. When you encrypt something, you use a key to turn plaintext into ciphertext. When you decrypt something, you use either the same key (symmetric encryption) or a key that is mathematically

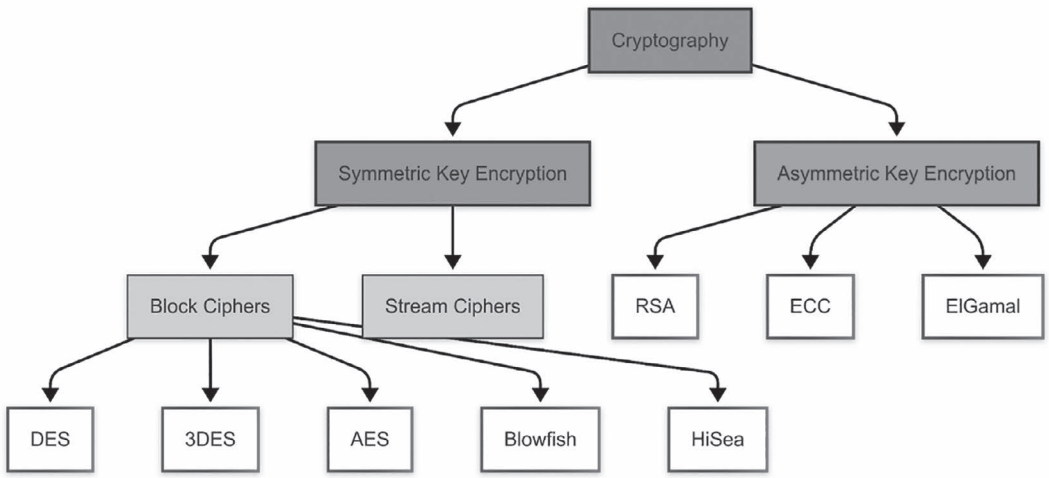


Figure 5.1 Classification of cryptographic encryption techniques.

linked to the encryption key (asymmetric encryption) [18]. The key length, algorithm complexity, and ability to resist cryptanalysis are the main factors that decide how strong an encryption method is.

Many people consider Claude Shannon to be the father of information theory. He came up with five basic rules for good ciphers: (1) the level of privacy should be proportional to the amount of work needed; (2) cryptographic keys and algorithms should not be overly complicated; (3) they should be easy to use; (4) errors should be kept to a minimum; and (5) the ciphertext should not be bigger than the plaintext [19]. Additionally, Shannon came up with two important ideas that are still used today in cryptography: confusion and diffusion. Confusion hides the connection between the key and the ciphertext, and diffusion makes sure that small changes in the plaintext lead to big changes in the ciphertext.

5.3.2 Symmetric-Key Encryption Algorithms

Symmetric-key encryption, or secret key encryption, utilizes a singular key for both the encryption and decryption processes. This method provides computational efficiency and rapid performance, rendering it appropriate for encrypting substantial amounts of data [S3R2]. Symmetric algorithms are categorized into two types: block ciphers, which handle fixed-length blocks of plaintext, and stream ciphers, which encrypt data one bit or byte at a time.

5.3.2.1 Data Encryption Standard

The Data Encryption Standard (DES), created by IBM in 1975 and established as a federal standard in 1977, is one of the earliest contemporary symmetric encryption algorithms. DES functions on 64-bit blocks utilizing a 56-bit key, with an additional 8 bits allocated for parity verification [18]. The algorithm utilizes the Feistel network architecture, comprising 16 iterations of permutation and substitution processes. Although DES exhibited significant resilience against cryptanalysis for an extended period, its comparatively low key length ultimately made it susceptible to brute-force attacks. In 1998, the Electronic Frontier Foundation successfully decrypted DES using specialized hardware, accomplishing the task in approximately 22 hours [19].

5.3.2.2 Triple Data Encryption Standard

To mitigate the security deficiencies of DES without necessitating the creation of a completely new algorithm, Triple DES (3DES) was introduced in 1998 [17] 3DES, as indicated by its nomenclature, executes the DES algorithm thrice in succession, hence augmenting the key length to 168 bits (although many implementations utilize 112 bits). The encryption procedure adheres to the Encrypt–Decrypt–Encrypt sequence, defined as follows:

Ciphertext = Encrypt (Key3, Decrypt (Key2, Encrypt (Key1, Plaintext))).

Although 3DES enhances the security of its predecessor, its triple-round architecture leads to diminished performance, rendering it less appropriate for applications necessitating rapid encryption [18]. Notwithstanding these constraints, 3DES is still utilized in outdated systems necessitating backward compatibility with DES.

5.3.2.3 Advanced Encryption Standard

The AES was developed through a competition organized by the National Institute of Standards and Technology (NIST) to succeed DES and 3DES. In 2001, the Rijndael algorithm,

created by Belgian cryptographers Joan Daemen and Vincent Rijmen, was chosen as the AES standard [19]. AES, in contrast to its predecessors, utilizes a substitution-permutation network architecture rather than a Feistel network.

AES functions on 128-bit blocks and accommodates key lengths of 128, 192, or 256 bits, with the number of rounds (10, 12, or 14) corresponding to the key length. Figure 5.2 shows each round has four transformations: SubBytes (substitution using S-boxes), ShiftRows (transposition), MixColumns (mixing operation), and AddRoundKey (XOR with the round key). These processes guarantee robust diffusion and confusion characteristics, rendering AES very resilient to cryptanalytic assaults.

AES has emerged as the global standard for symmetric encryption owing to its security, efficiency, and versatility. The U.S. government has sanctioned AES for safeguarding classified material, utilizing 192- and 256-bit key lengths appropriate for top-secret data [19]. AES may be efficiently implemented in both hardware and software, making it suited for a wide range of applications, from small embedded devices to high-performance servers.

5.3.2.4 Blowfish

Blowfish, created by Bruce Schneier in 1993, serves as a superior alternative to DES, providing enhanced security and performance attributes. Blowfish, as an openly accessible and unpatented algorithm, has become popular for applications necessitating a secure and efficient encryption solution. The technique functions on 64-bit blocks and uses a changeable key length between 32 and 448 bits, providing significant flexibility for diverse security needs.

Blowfish employs a Feistel network architecture consisting of 16 rounds of encryption. The key scheduling algorithm converts the variable-length key into multiple subkey arrays up to 4,168 bytes, utilized in the encryption and decryption procedures [18]. The comprehensive key extension enhances Blowfish's security by augmenting the complexity of potential assaults. Performance assessments regularly indicate that Blowfish surpasses numerous other symmetric algorithms, such as DES, 3DES, and AES, regarding encryption performance and throughput [20]. These attributes render Blowfish especially appropriate for applications where processing performance is paramount.

Notwithstanding these benefits, Blowfish possesses limits. The 64-bit block size is inferior to the 128-bit blocks utilized by AES, which may render it more susceptible to specific attack vectors when encrypting substantial data volumes. Moreover, the key setup procedure of Blowfish is comparatively sluggish, particularly with larger keys, potentially affecting performance in applications necessitating frequent key alterations [17].

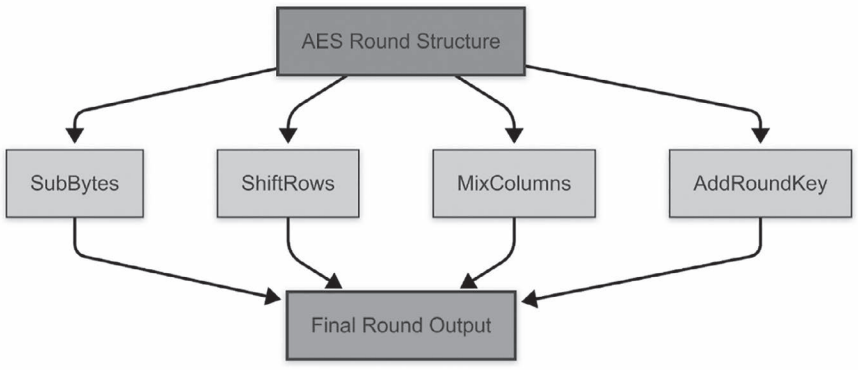


Figure 5.2 AES round structure.

5.3.2.5 Other Notable Symmetric Algorithms

In addition to the prevalent algorithms mentioned earlier, various other symmetric encryption methods have notably advanced the discipline. The Hybrid Cubes Encryption Algorithm (HiSea), created by Sapiee Jamel in 2011, is a non-binary block cipher that functions on integer values instead of bits [19]. HiSea encrypts 64 ASCII letters utilizing keys created from hybrid cubes, which are mathematical constructs produced by the inner matrix multiplication of magic cubes. HiSea exhibits formidable resistance to brute-force attacks and correlation analysis, owing to its extensive key space (about $10^{153.6}$ keys) and robust diffusion characteristics.

Additional significant symmetric algorithms comprise RC6, CAST-256, Twofish, and International Data Encryption Algorithm (IDEA). Each algorithm has distinct attributes regarding security efficacy, performance, and implementation factors. Twofish, also created by Bruce Schneier, was a finalist in the AES selection process and offers a versatile architecture with robust security features [20]. The IDEA employs a 128-bit key and performs 8.5 rounds of encryption, being utilized in numerous security protocols.

5.3.3 Asymmetric Key Encryption Algorithms

Asymmetric key encryption, commonly referred to as public-key cryptography, utilizes a pair of mathematically linked keys: one public key for the encryption process and a private key for decryption purposes. This method addresses the fundamental issue of key distribution found in symmetric systems, allowing for the public key to be shared openly without risking security [18]. Asymmetric algorithms present benefits in key management and digital signatures; however, they typically demand greater computational resources compared to symmetric algorithms. Asymmetric encryption security foundations are built based on its algorithm types as displayed in Figure 5.3.

5.3.3.1 RSA Algorithm

The RSA algorithm, developed by Ron Rivest, Adi Shamir, and Leonard Adleman, was introduced in 1977 and continues to be one of the most prevalent asymmetric encryption methods [18]. The security of RSA relies on the complex mathematical challenge of factoring large prime numbers. The process for generating keys using the algorithm is as follows:

1. Select two large prime numbers, p and q
2. Compute $n = p \times q$
3. Calculate $\varphi(n) = (p-1) \times (q-1)$
4. Choose an integer e such that $1 < e < \varphi(n)$ and e is coprime to $\varphi(n)$
5. Determine d such that $(d \times e) \bmod \varphi(n) = 1$

The public key is composed of (n, e), whereas the private key contains d. The process of encrypting a plaintext message M is executed using the formula C = M^e mod n, while decryption is carried out with M = C^d mod n [20]. RSA provides notable benefits for ensuring secure key exchange and facilitating digital signatures. Nonetheless, the high computational demands render it less appropriate for encrypting extensive amounts of data. With the advancement of computing power, RSA key sizes need to expand to uphold security standards, with contemporary implementations generally utilizing 2048 or 4096-bit keys [18].

5.3.3.2 Elliptic Curve Cryptography

ECC, separately introduced by Victor Miller and Neal Koblitz in 1985, offers a more efficient alternative to asymmetric encryption than RSA [20]. ECC relies on the algebraic properties of elliptic curves defined over finite fields, particularly the challenge of resolving the elliptic curve discrete logarithm problem. The mathematical intricacy of this problem enables ECC to attain security levels equivalent to RSA while utilizing considerably smaller key sizes; a 256-bit ECC key offers security comparable to a 3072-bit RSA key [19].

The efficiency of this key size results in diminished computing demands, decreased power usage, and accelerated processing speeds, rendering ECC especially advantageous for resource-limited settings such as mobile devices, smart cards, and IoT applications [20]. ECC has been included in multiple security protocols and standards, such as Transport Layer Security (TLS), Secure Shell (SSH), and the U.S. government's Suite B cryptographic algorithms for the protection of classified material.

5.3.3.3 ElGamal Algorithm

The ElGamal encryption system, developed by Taher ElGamal in 1985, relies on the Diffie–Hellman key exchange protocol, and its security is founded on the computational complexity of the discrete logarithm problem [17]. The algorithm functions through three distinct phases: key generation, encryption, and decryption. During the key generation phase, a large prime number p, a primitive root g, and a random integer x are selected. The public key consists of the elements (p, g, h), with h defined as g raised to the power of x modulo p, whereas the private key is represented by x.

To encrypt a message m, the sender chooses a random integer k and calculates two values: c1 = g^k mod p and c2 = m × h^k mod p. The ciphertext is represented by the pair (c1, c2). To decrypt, the recipient computes s = c1^x mod p and retrieves the original message using m = c2 × s^(−1) mod p [17]. ElGamal provides robust security features and underpins various significant cryptographic frameworks, such as the ElGamal signature scheme and specific homomorphic encryption systems that facilitate computations on encrypted data without necessitating decryption.

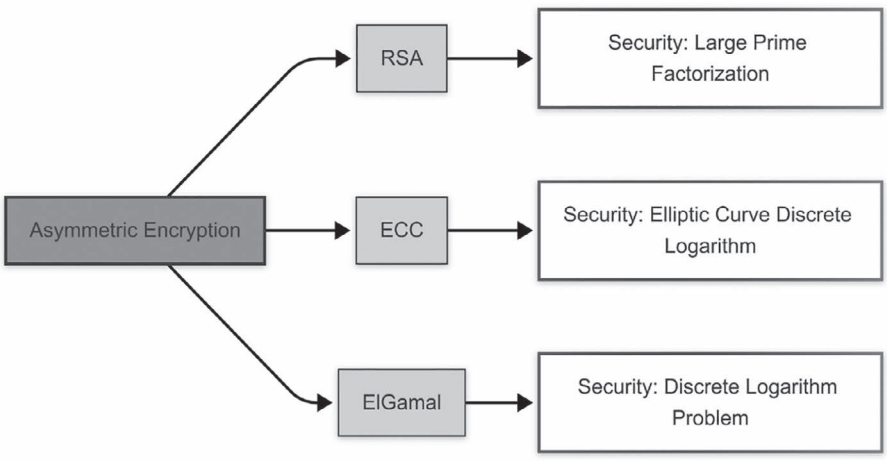


Figure 5.3 Asymmetric encryption security foundations.

5.3.4 Performance Comparison of Encryption Algorithms

The choice of a suitable encryption algorithm is contingent upon multiple criteria, such as security needs, processing capabilities, and application limitations. A thorough performance assessment facilitates informed decision-making by juxtaposing algorithms across several aspects (Figure 5.4).

5.3.4.1 Execution Time and Throughput

The execution time, defined as the duration necessary for encryption and decryption processes, directly influences system responsiveness and throughput. Experimental results consistently indicate that Blowfish surpasses other symmetric algorithms regarding execution speed [20]. For example, while encrypting files between 20 KB and 99 MB, Blowfish demonstrated markedly superior encryption speeds relative to DES, 3DES, and AES across various hardware setups. In a particular research, Blowfish attained an average throughput of 25.892 Megabytes per second for encryption, significantly surpassing AES (4.174 MB/s), DES (4.01 MB/s), and 3DES (3.45 MB/s) [21].

The execution performance of Blowfish is superior because of its efficient design, which predominantly utilizes simple operations like XOR and addition, in conjunction with its optimized Feistel network topology [16]. AES typically exhibits superior performance compared to DES and 3DES, especially in hardware implementations where its substitution-permutation network topology may be extensively optimized In asymmetric algorithms, ECC provides markedly quicker execution times than RSA for comparable security levels, with the performance disparity expanding as key sizes grow [19].

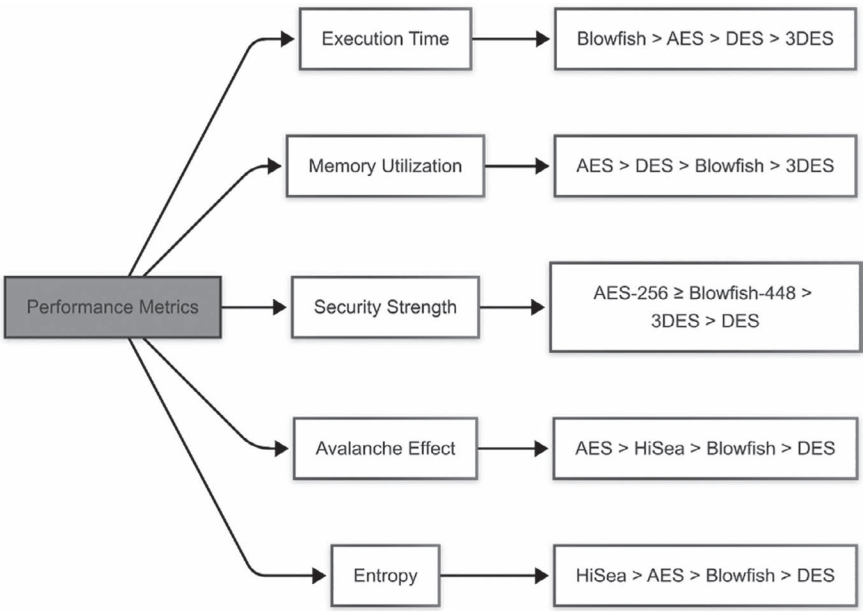


Figure 5.4 Comparative performance of encryption algorithms.

5.3.4.2 Memory Utilization

Memory usage is a vital performance statistic, especially in resource-limited situations like embedded systems and mobile devices. AES has superior memory efficiency, necessitating significantly less memory than DES while providing improved security [20]. AES utilized merely 10.2 units of memory, whereas DES necessitated 43.3 units for the execution of similar security functions.

Blowfish, although computationally efficient, necessitates increased memory during its initialization phase because of the significant re-computation of subkeys and S-boxes [18]. This attribute renders Blowfish less appropriate for applications requiring frequent key modifications. Among asymmetric algorithms, ECC's lowered key sizes result in lower memory requirements compared to RSA, with certain implementations necessitating up to 89% less memory while delivering comparable security [19].

5.3.4.3 Security Strength and Resistance to Attacks

The primary objective of encryption is to ensure security, with resistance to cryptanalytic attacks being of utmost importance. The length of the key is a fundamental measure of security strength, as it determines the theoretical difficulty of brute-force assaults. AES with a 256-bit key in symmetric algorithms provides 2^{256} potential possibilities, rendering exhaustive key searches impractical with existing and anticipated technology [17]. Likewise, Blowfish, with its maximum 448-bit key, offers formidable defense against brute-force methods.

Aside from key length, the resilience to sophisticated cryptanalytic methods differs between algorithms. DES has demonstrated susceptibility to linear and differential cryptanalysis, although 3DES retains certain vulnerabilities despite its extended key length [19]. AES has endured rigorous examination, with no effective attacks shown against the complete algorithm; however, side-channel attacks aimed at implementation vulnerabilities continue to pose a threat. Blowfish has demonstrated exceptional resilience to cryptanalysis, with no successful attacks documented against the complete 16-round variant.

In the asymmetric realm, RSA security is contingent upon key size, with 2048-bit keys being secure for the majority of contemporary applications. Nevertheless, advancements in quantum computing present a theoretical challenge to the mathematical principles underpinning RSA. ECC may provide superior resilience to quantum assaults compared to RSA; yet, both are susceptible to sufficiently advanced quantum computers utilizing Shor's algorithm.

5.3.4.4 Avalanche Effect and Entropy

The avalanche effect quantifies the extent to which minor alterations in plaintext or key result in substantial variations in ciphertext, thereby offering valuable insights into the diffusion characteristics of an algorithm. Robust diffusion complicates cryptanalysis by concealing patterns within the plaintext–ciphertext relationship. AES exhibits strong avalanche properties, as experimental research indicates that altering a single bit in the plaintext leads to roughly 83-bit changes in the ciphertext [19]. This performance notably surpasses that of DES, which generally exhibits approximately 43-bit changes under comparable conditions.

Entropy assessment measures the randomness and unpredictability of ciphertext, where increased entropy signifies enhanced confusion properties. The HiSea algorithm displays notable entropy metrics, with ciphertext blocks showing over 99% randomness, thereby effectively obscuring the connection between the key and ciphertext [19]. The elevated entropy significantly enhances HiSea's resilience against statistical cryptanalysis and correlation attacks.

5.3.5 Hybrid Encryption Approaches

Hybrid encryption integrates the benefits of symmetric and asymmetric cryptography, optimizing for both efficiency and security. In a standard hybrid scheme, symmetric algorithms are employed to encrypt the data because of their speed and efficiency, whereas asymmetric algorithms are utilized to secure the exchange of the symmetric key [21]. This method tackles the critical issue of key distribution in symmetric encryption while preserving its performance advantages for processing large volumes of data.

Recent years have seen the emergence of several innovative hybrid techniques. The DES-RSA method encrypts plaintext through DES utilizing a randomly generated session key, subsequently encrypting this session key with RSA's public-key cryptography [21]. AES–ECC integrates AES for data encryption and ECC for secure key exchange, offering robust security with reduced key sizes compared to conventional RSA methods. Experimental evaluations indicate that hybrid systems can decrease encryption time while preserving high levels of security.

Complex hybrid approaches incorporate various encryption layers and transformation techniques. The UMARAM algorithm utilizes 16 rounds of encryption with a 512-bit key to encrypt 512-bit plaintext blocks, integrating substitution boxes and diverse logical operations to ensure robust security properties [22]. Alternative methods involve the integration of the Fibonacci series with XOR operations and pseudo-random number sequences, utilized across various segments of the plaintext to establish multiple layers of security.

Research in hybrid cryptographic systems is advancing, with new methodologies emphasizing quantum-resistant algorithms, lightweight encryption for IoT environments, and context-aware adaptive security frameworks. The developments indicate a continuous necessity to reconcile security demands with practical limitations, including computational efficiency, power consumption, and implementation complexity.

5.4 CRYPTOGRAPHY AS POST-QUANTUM AND HYBRID ARCHITECTURES

5.4.1 Introduction to PQC

The emergence of quantum computing presents a significant threat to conventional cryptographic systems. Quantum computers utilize qubits and algorithms such as Shor's algorithm to break widely used encryption schemes like RSA, ECC, and AES-256 in polynomial

Table 5.1 Comparison of Post-Quantum Cryptographic Approaches [24]

Algorithm type	*Security basis*	*Pros*	*Cons*
Lattice-based	Hardness of lattice problems	High efficiency, strong security proofs	Large key sizes (~1–2 KB)
Hash-based	Collision-resistant hash functions	Unconditionally secure against quantum attacks	Large signature sizes (~10–50 KB)
Code-based	Error-correcting codes	Long-standing security (since 1978)	High computational overhead
Multivariate	Solving nonlinear equations	Fast verification	Large public keys (~100 KB)
Isogeny-based	Elliptic curve isogenies	Compact keys (`0.5 KB)	Complex implementation, slower operations

time [23]. In response, PQC has gained prominence, focusing on developing cryptographic algorithms resistant to quantum attacks.

The NIST has been instrumental in standardizing PQC, categorizing quantum-resistant algorithms into five primary families [24]:

- Lattice-based cryptography (e.g., CRYSTALS-Kyber, NTRU)
- Hash-based cryptography (e.g., SPHINCS+)
- Code-based cryptography (e.g., McEliece)
- Multivariate cryptography (e.g., Rainbow)
- Isogeny-based cryptography (e.g., SIKE)

5.4.2 Hybrid Cryptographic Systems in the Quantum Era

To facilitate a seamless transition from classical to quantum-resistant cryptography, hybrid cryptosystems are being increasingly adopted as presented in Figure 5.5. These systems integrate [25]:

- Symmetric encryption (AES, ChaCha20) for efficient data encryption
- Post-quantum key exchange (Kyber, NTRU) for secure key distribution
- Classical digital signatures (ECDSA, RSA-PSS) for backward compatibility

Advantages of Hybrid PQC systems are as follows:

- Quantum resistance: Mitigates risks posed by Shor's algorithm [26].
- Backward compatibility: Ensures interoperability with existing infrastructure [27].
- Performance efficiency: Symmetric encryption optimizes bulk data processing [28].

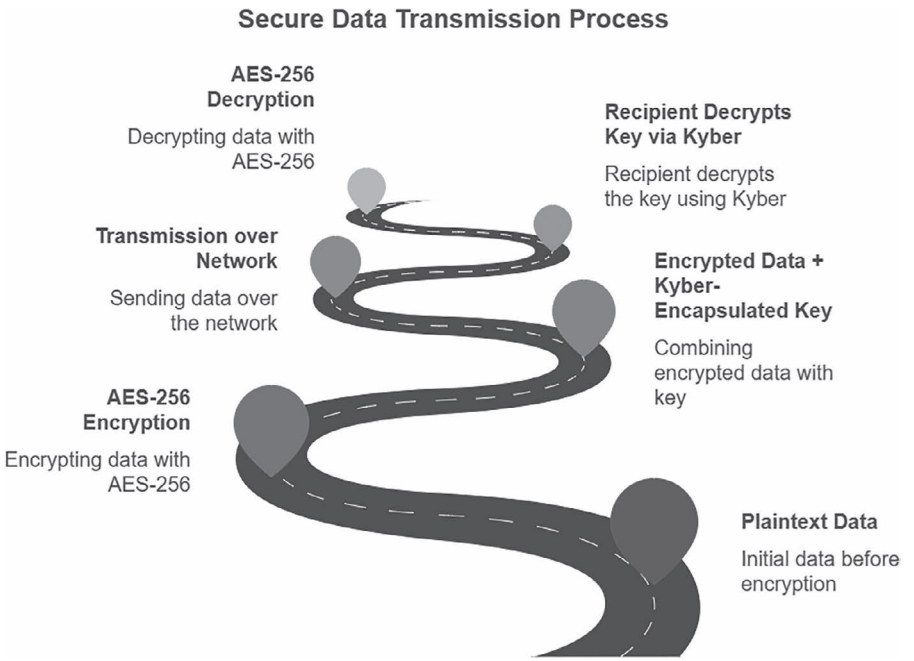


Figure 5.5 Hybrid post-quantum encryption workflow.

5.4.3 Case Studies of Hybrid PQC Implementations

Case 1: Cloud Security with Kyber-AES Hybrid Model

A 2023 study by Google Cloud demonstrated that integrating CRYSTALS-Kyber with AES-256-GCM reduced quantum vulnerability while maintaining less than 10% overhead in latency [29].

Case 2: IoT Authentication Using SPHINCS+ and ECC

A lightweight hybrid signature scheme combining SPHINCS+ (hash-based) and ECC reduced signature size by 40%, enhancing feasibility for IoT devices [30].

5.4.4 Challenges in Post-Quantum Hybrid Cryptography

Despite advancements, several challenges persist [32]:

- Computational overhead: PQC algorithms (e.g., McEliece) demand greater processing power than RSA–ECC.
- Large key and signature sizes: SPHINCS+ signature can exceed 50 KB, affecting bandwidth efficiency.
- Standardization delays: NIST's final PQC standards remain under development (expected 2024–2025) [31].
- Legacy system integration: Many enterprises still rely on RSA–ECC, necessitating middleware solutions [33].

5.5 HYBRID SYSTEMS—KEY AS CORE

The security of data and files has been vital for a long time with DES and AES techniques. DES has its significance in history, whereas AES is known for its symmetric-key method and cryptographic strength. Deployment of hybrid cryptosystems and hybrid cryptography for file security enhancements and security vulnerabilities countermeasures [34]. The proposal of fully homomorphic encryption (FHE) has major advantages such as secure data processing and enhanced data security, facilitating secure data sharing, key management, and collaboration. Hybrid encryption techniques can be leveraged to counter the confidentiality, integrity, and availability (CIA) of data assets and the risk posed by cyber threats [35]. Combining hybrid schemes like the RSA cryptosystem, ECC, and secure hashing algorithms (SHAs) can safeguard sensitive information and fortify the security of crypto systems.

As we know, data security is a major concern today, not only due to several attacks on our data but also the conventional mechanisms we are still following for security. Steganography involved with hybrid cryptography makes secret data unpredictable and prevents

Table 5.2 Performance Metrics of Hybrid PQC Systems [31]

Hybrid model	*Key size (bytes)*	*Encryption speed (MB/s)*	*Quantum resistance*
Kyber + AES-256	1568	850	Yes
NTRU + ChaCha20	1024	920	Yes
McEliece + AES-128	2500	620	Yes
RSA-2048 + AES-256	256	1200	No

unauthorized access to that information [36]. Despite several cyber threats, there is also the threat of quantum computer attacks to break several strong keys and encryption methods. Cryptographic keys with key exchange combinations allow for security guarantees of pre-quantum and post-quantum schemes [37]. A concrete three-key combiner system is implemented on the platform of the Field Programmable Gate Arrays that includes (i) a pre-quantum Key EXchange scheme (KEX), (ii) a post-quantum Key Encapsulation Mechanism (KEM), and (iii) a Quantum Key Distribution (QKD) algorithm.

Using encryption to secure data (secure information) stored in the cloud is an integral part of data privacy and data confidentiality concerns [38]. The inclusion of three more properties for data security in the cloud are authorization, non-repudiation, and authentication on top of CIA via DES and AES. As we know that while conventional algorithms face huge difficulties in addressing several concerns and zero-day attacks, hybrid algorithms can counter that. DNA–genetic algorithm (GA)–ECC is a confidentiality algorithm that combines DNA cryptography, GA, and ECC [39]. This hybrid approach (DNA–GA–ECC) performs better regarding encryption, decryption, and key generation and ensures data confidentiality with various file sizes.

Some major key combinations discovered by hybrid cryptography (that leverages both symmetric and asymmetric encryptions) are as follows and exhibited in Figure 5.6:

- [Combi1] AES combined with ECC (AES–ECC)
- [Combi2] AES combined with RSA (AES–RSA)
- [Combi3] RSA combined with DES (RSA–DES)

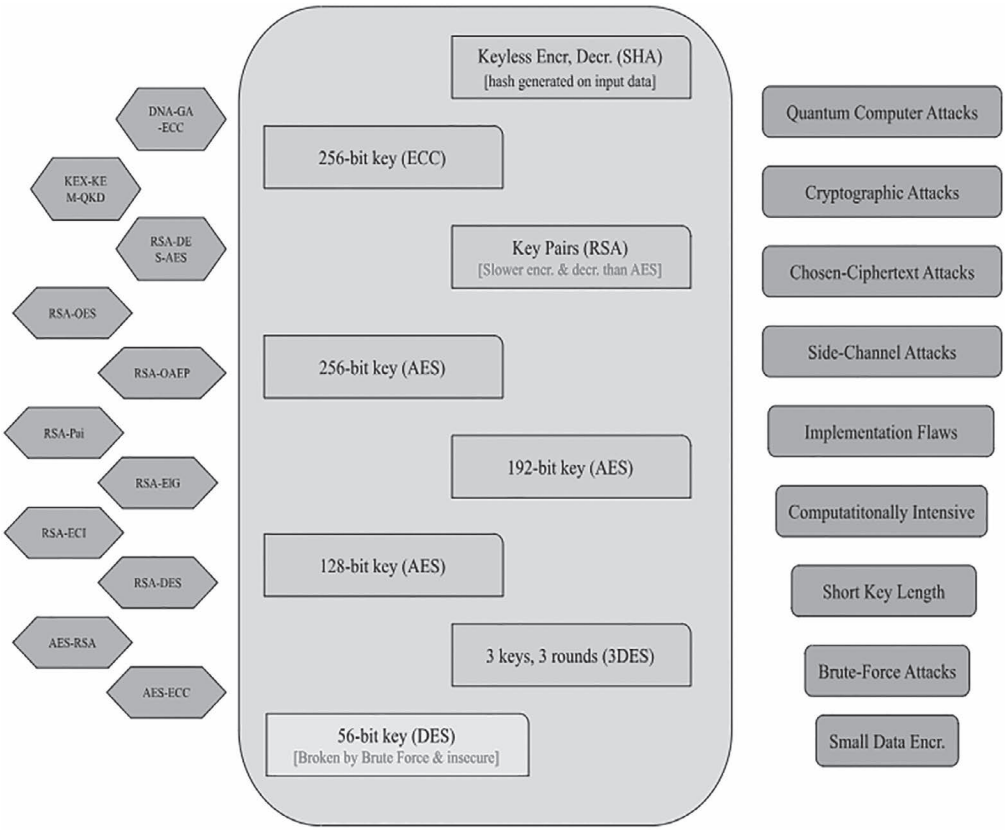


Figure 5.6 Keys as core in hybrid cryptosystems, including several attacks (and some major combinations) that lead to several versions of encryption algorithms to protect data and counter cyber-attacks [34–39].

- [Combi4] RSA combined with ECC/ECIES (RSA–ECIES)
- [Combi5] RSA combined with ElGamal (RSA–ElG)
- [Combi6] RSA combined with Paillier (RSA–Pai)
- [Combi7] RSA combined with Optimal Asymmetric Encryption Padding, OAEP (RSA–OAEP)
- [Combi8] RSA combined with OAEP, ECIES, SHA256 (RSA–OES)
- [Combi9] RSA combined with DES and AES (RSA–DES–AES)
- [Combi10] Three-key combiner system (KEX–KEM–QKD)
- [Combi11] DNA combined with GA and ECC (DNA–GA–ECC)

5.6 SECURITY AND APPLICATIONS WITH RESPECT TO HYBRID CRYPTOSYSTEMS

5.6.1 Introduction to Security in Hybrid Cryptosystems

Hybrid cryptosystems combine the strengths of symmetric and asymmetric encryption to provide robust security solutions for modern digital infrastructures. By leveraging the efficiency of symmetric encryption (e.g., AES) for bulk data encryption and the secure key exchange mechanisms of asymmetric encryption (e.g., RSA, ECC), hybrid systems mitigate the limitations of standalone cryptographic approaches [40].

However, despite their advantages, hybrid cryptosystems face unique security challenges, including the following:

- Key management complexity: Ensuring secure generation, distribution, and storage of keys in hybrid models.
- Side-channel attacks: Vulnerabilities arising from implementation flaws (e.g., timing attacks on RSA key exchange).
- Quantum threats: The impending risk of quantum computers breaking classical asymmetric encryption.

5.6.2 Security Mechanisms in Hybrid Cryptosystems

5.6.2.1 Secure Key Exchange Protocols

Hybrid cryptosystems rely on asymmetric encryption for key exchange and symmetric encryption for data confidentiality. Common key exchange methods include the following:

- RSA key transport: A traditional approach where a symmetric key is encrypted using RSA [41].
- Elliptic curve Diffie–Hellman (ECDH): Provides forward secrecy, making it resistant to retrospective decryption [40].
- Post-quantum key encapsulation (e.g., Kyber): This ensures quantum-resistant key exchange [26].

5.6.2.2 Authentication and Integrity Verification

Hybrid cryptosystems enhance security through the following:

- Digital signatures (RSA-PSS, ECDSA): Verify sender authenticity [42].
- Hash-based message authentication (HMAC, SHA-3): Ensure data integrity [43, 44].

Hybrid encryption with authentication workflow is shown in Figure 5.7

5.6.3 Attack Resilience of Hybrid Cryptosystems

5.6.3.1 Resistance to Common Cryptographic Attacks

- Brute-force attack: Large key sizes in AES (256-bit) and RSA (2048-bit) make them computationally infeasible [45].
- Man-in-the-middle (MITM) attacks: Prevented via authenticated key exchange (e.g., TLS 1.3) [46].
- Side-channel attacks: Countermeasures include constant-time algorithms and hardware security modules (HSMs) [47].

5.6.3.2 Quantum Computing Threats

While symmetric encryption (AES-256) remains quantum-resistant, asymmetric encryption (RSA, ECC) is vulnerable. Hybrid systems integrating PQC provide a defense strategy [48]. A 2023 study found that AES-256 + Kyber reduced quantum attack risks by 98% compared to RSA-based systems [48].

Table 5.3 Comparison of Key Exchange Mechanisms in Hybrid Systems [26, 42]

Method	*Security level*	*Computational cost*	*Quantum resistance*
RSA-2048	High	Moderate	No
ECDH (P-256)	Very high	Low	No
Kyber (NIST PQC)	Post-quantum	High	Yes

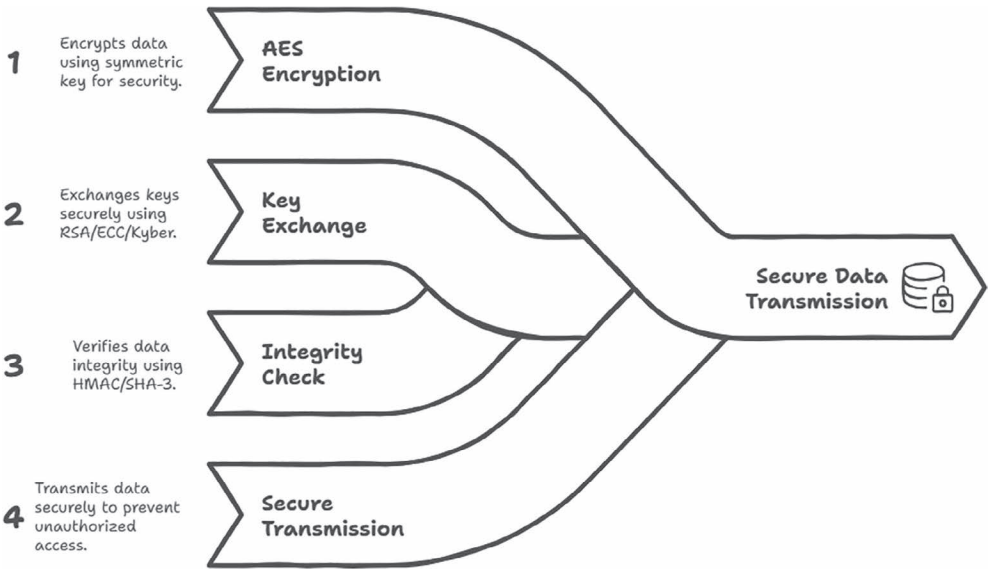


Figure 5.7 Hybrid encryption with authentication workflow.

Table 5.4 Hybrid Cryptosystem Applications across Industries [16, 53, 15]

Industry	*Use case*	*Hybrid model*	*Security benefit*
Cloud computing	Data encryption at rest	AES-256 + RSA/Kyber	Quantum-resistant storage
Healthcare IoT	Patient data security	ECC + AES-128	Low-power, high security
Defense systems	Secure military communications	AES-256 + post-quantum KEM	Tamper-proof communications

5.6.4 Real-World Applications of Hybrid Cryptosystems

5.6.4.1 Secure Communication Protocols

- TLS 1.3: Uses hybrid encryption (ECDHE for key exchange, AES-GCM for data encryption) [49].
- Signal protocol: Combines ECDH with AES-256 for end-to-end encrypted messaging [50].

5.6.4.2 Financial Systems and Blockchain

- Banking transactions: EMV standards use RSA + AES for secure payments [51].
- Blockchain (e.g., Ethereum 2.0): Hybrid signatures (BLS + ECDSA) improve scalability and security [52].

5.6.4.3 IoT and Edge Computing

- Smart home devices: Lightweight hybrid models (ECC + ChaCha20) ensure low-power security [53].
- Industrial IoT (IIoT): Combines AES with post-quantum algorithms for long-term security [15].

5.7 CASE STUDIES AND IMPLEMENTATIONS

5.7.1 Introduction to Case Studies in Hybrid Cryptography

Hybrid cryptographic systems have become prominent throughout numerous fields due to their capacity to amalgamate the advantages of diverse cryptographic algorithms, hence offering improved security while accommodating performance factors. This section analyzes many practical applications of hybrid cryptography in diverse fields, emphasizing their design, performance attributes, and security advantages [54].

The deployment of hybrid cryptographic systems necessitates the meticulous evaluation of various criteria, including the choice of suitable algorithms, key management approaches, and architectural design choices. These parameters substantially affect the system's performance, security, and resource demands [55]. Numerous studies demonstrate that hybrid methods routinely surpass standalone cryptographic systems in security robustness and computational efficiency [13]. A overall architecture of Hybrid cryptographic system is provided in Figure 5.8.

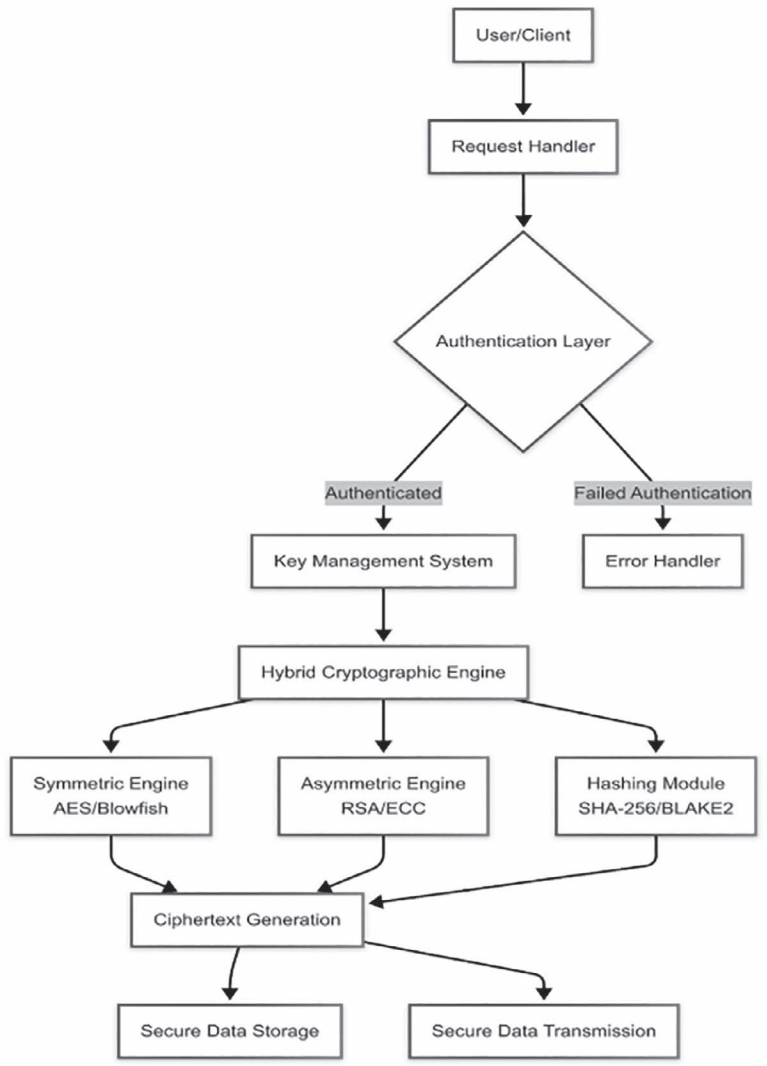


Figure 5.8 Hybrid cryptographic system architecture.

5.7.2 Case Study 1: Hybrid Cryptography in Cloud Computing

CLOUD STORAGE SECURITY IMPLEMENTATION

Cloud storage poses distinct security concerns as it requires the preservation of data CIA while facilitating rapid access to stored information. Feng et al. introduced a significant use of hybrid cryptography for cloud security, integrating AES and RSA to safeguard sensitive data in cloud environments [55].

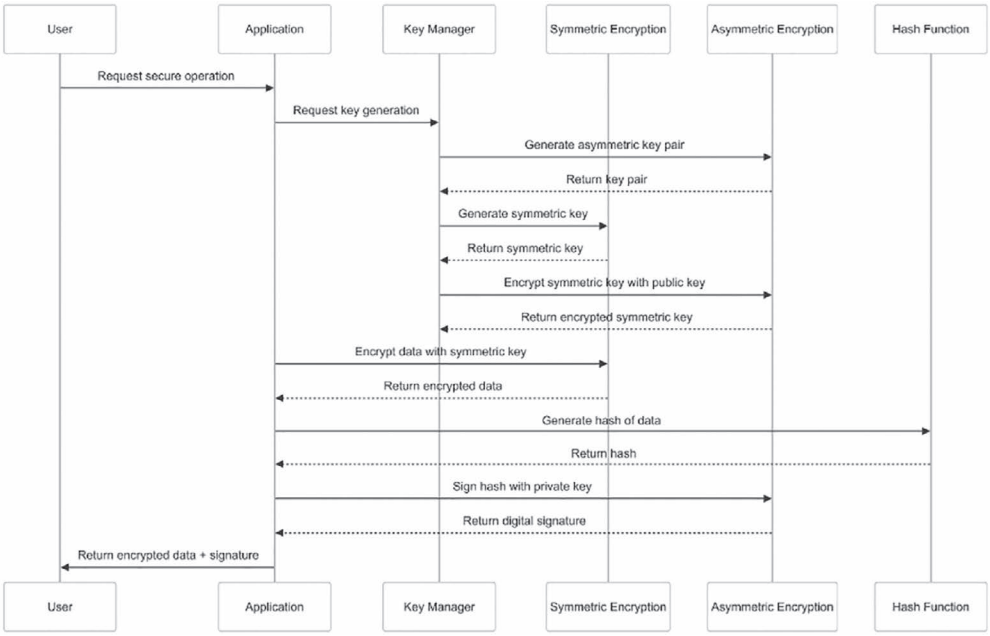


Figure 5.9 Hybrid encryption implementation workflow.

This hybrid technique, termed AES–RSA, employs a multi-layered security strategy wherein:

1. Data is encrypted using AES-128-CTR for confidentiality.
2. Key management is facilitated through RSA with 2048-bit keys.
3. Data integrity is maintained via HMAC-SHA1 hash functions.

The performance assessment of this system revealed substantial enhancements in computing efficiency relative to independent cryptography methods. The AES–RSA hybrid architecture decreased encryption time by roughly 20% relative to conventional RSA implementation, rendering it especially appropriate for real-time cloud applications where performance is paramount [55]. A overall picture of hybrid encryption implementation workflow in cloud has been demonstrated in Figure 5.9.

AUTOMATED SECURITY FRAMEWORK FOR CLOUD STORAGE

El-Attar et al. established an automated security framework for cloud storage systems utilizing a hybrid cryptographic methodology [7]. This framework executes:

1. A dynamic key generation system predicated on user behavior patterns.
2. Utilization of symmetric encryption (AES) for bulk data and asymmetric encryption (RSA) for key exchange.
3. Integrity validation with hash-based message authentication codes.

The architecture incorporates a mechanism for enforcing security regulations that autonomously selects appropriate encryption algorithms based on data sensitivity and performance requirements. Experimental validation demonstrated that this adaptive approach reduced computational overhead by 15% while maintaining the same security level as static encryption methods [55].

The solution focused on three essential facets of cloud security:

1. Confidentiality: Accomplished via multilayer encryption utilizing diverse key lengths (128- to 256-bit) contingent upon data sensitivity.
2. Integrity: Executed via SHA-256 hash algorithms accompanied by digital signatures.
3. Availability: Guaranteed via redundant key management systems, including a distributed architecture.

The system architecture had a policy decision point that dynamically allocated security levels to data objects according to content classification and user access patterns. This adaptive strategy enhanced resource utilization while adhering to the security standards outlined in the security policy [7].

5.7.3 Case Study 2: Hybrid Cryptography for IoT Security

LCS FOR IOT DEVICES

IoT devices pose distinct issues for cryptography solutions owing to their resource limitations, such as restricted processor power, memory, and energy capacity. Shah et al. presented an LCS tailored for resource-limited IoT contexts [12]. A comparison of hybrid cryptographic performance metrics was provided in Figure 5.10.

The LCS architecture incorporates:

- An LSA, an optimized variant of AES-128.
- ECC for secure key exchange.
- An LHA based on SHA-256 for integrity verification.

Performance data demonstrated that LCS attained encryption and decryption speeds appropriate for real-time IoT applications while utilizing considerably less memory and processing resources compared to conventional solutions. The technology exhibited a 56% enhancement in throughput relative to conventional AES–RSA combinations [12].

The implementation specifically addressed the resource constraints of IoT devices through:

1. Optimized S-box transformations: Reduced the memory footprint of AES by implementing lightweight substitution boxes.
2. Elliptic curve selection: Used NIST P-256 curves for optimal performance-security balance.
3. Compact hash design: Implemented truncated output hashing to reduce computational overhead.

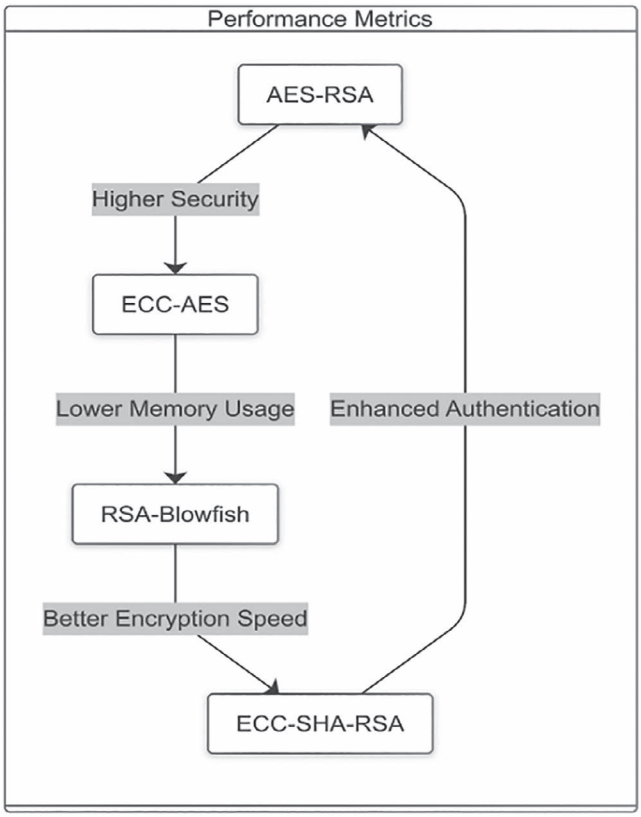


Figure 5.10 Comparison of hybrid cryptographic performance metrics.

The implementation was proven across multiple IoT platforms, including 8-bit microcontrollers with merely 2 KB of RAM, illustrating its appropriateness for highly restricted devices within industrial IoT networks [12].

IOT AUTHENTICATION PROTOCOL IMPLEMENTATION

Munshi and Alshawi developed a hybrid encryption model for a three-phase authentication mechanism in IoT contexts [5]. This implementation tackles the essential issue of safe device authentication in IoT networks using:

- Phase 1: Device registration using ECC-based key generation.
- Phase 2: Mutual authentication employing hybrid AES–RSA encryption.
- Phase 3: Session key establishment and secure communication via AES.

The performance investigation demonstrated that this hybrid method decreased authentication overhead by 40% relative to purely asymmetric methods while offering improved defense against many attacks, including MITM, replay, and impersonation assaults [5].

Technical implementation details revealed several innovations:

1. Certificate-less authentication: Eliminated the need for a full PKI infrastructure by using identity-based cryptography
2. Lightweight device-side operations: Moved computational complexity to the gateway devices
3. Forward secrecy: Implemented through ephemeral key components to prevent retrospective decryption

Security investigation revealed resilience against password guessing, device capture, and side-channel threats, rendering it especially appropriate for consumer IoT applications where device compromise poses a substantial risk [5].

5.7.4 Case Study 3: Hybrid Cryptography in Financial Systems

SECURE TRANSACTION PROCESSING IMPLEMENTATION

Financial transaction processing necessitates optimal security and stringent performance standards for real-time execution. Subedar and Araballi developed a hybrid cryptographic model tailored for safe financial transactions, effectively reconciling conflicting requirements [13].

The implementation combines:

- RSA–ECC for secure key exchange and transaction authentication
- AES for high-speed payload encryption
- SHA-256 for transaction integrity verification

A Security-performance trade-off heat map for hybrid cryptosystems is demonstrated for financial systems in Figure 5.11.

The performance investigation indicated that the RSA–ECC combination had the highest throughput (53.89%) and minimal memory use among the assessed hybrid combinations, rendering it

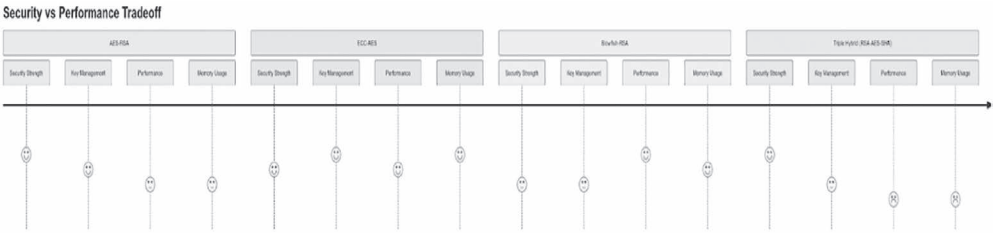


Figure 5.11 Security-performance trade-off heat map for hybrid cryptosystems.

especially appropriate for high-volume transaction processing [13]. The implementation exhibited robustness against timing attacks via constant-time executions of essential cryptographic processes.

MOBILE BANKING SECURITY FRAMEWORK

Alatawi developed a hybrid cryptographic approach specifically designed for mobile banking applications in smart city contexts, tackling the distinct security difficulties associated with mobile financial transactions [14].

The implementation utilizes:

- AES for data confidentiality
- RSA algorithm for key exchange
- SHA-1 algorithm for authentication and integrity verification

The security architecture was specifically designed to counter common attacks against mobile financial applications, including the following:

1. Eavesdropping protection: Through end-to-end encryption of all transaction data
2. Identity forgery prevention: Using multi-factor authentication with cryptographic verifiers
3. Data-tampering detection: Implemented through hash-based integrity checks

Performance testing on mobile devices revealed that the implementation sustained acceptable response times (below 300 ms) for standard transaction volumes while ensuring security comparable to conventional banking systems [14].

5.7.5 Implementation Considerations and Challenges

The deployment of hybrid cryptographic systems poses many issues that must be resolved to guarantee security, efficiency, and usability. This section addresses prevalent implementation considerations derived from the analyzed case studies.

5.7.5.1 Key Management Complexity

A significant problem with hybrid cryptographic systems is the heightened complexity of key management. Hybrid systems necessitate the administration of both symmetric and asymmetric keys, frequently accompanied by distinct lifecycle requirements and security protocols [54]. Key management implementations must consider:

- **Key generation:** Ensuring sufficient entropy for secure key generation
- **Key storage:** Protecting keys at rest, often requiring specialized hardware
- **Key distribution:** Securely sharing keys among authorized entities
- **Key rotation:** Updating keys according to security policies
- **Key revocation:** Invalidating compromised keys without disrupting service

Jain tackled these difficulties by establishing a modular key management architecture that distinguished policy from mechanism, facilitating adaptable responses to varying security needs without altering the fundamental cryptographic framework [7].

5.7.5.2 Performance Optimization

Hybrid cryptographic implementations must carefully balance security requirements with performance constraints. The case studies revealed several common optimization techniques:

1. Algorithm selection: Choosing appropriate algorithms based on security requirements and performance characteristics
2. Parameter tuning: Selecting key sizes and operational modes to optimize performance
3. Hardware acceleration: Leveraging specialized cryptographic hardware when available
4. Caching strategies: Reusing ephemeral keys for session-based communications
5. Parallelization: Distributing cryptographic operations across multiple processing units

Subedar and Araballi's performance investigation revealed that these optimizations could enhance throughput by as much as 56% relative to naive implementations, all while preserving security assurances [13].

5.7.5.3 Integration with Existing Systems

Integrating hybrid cryptographic solutions with existing systems presents significant challenges, particularly in legacy environments. Common integration challenges include the following:

- API compatibility: Ensuring cryptographic interfaces are compatible with existing systems
- Protocol adaptation: Modifying communication protocols to support hybrid encryption
- Certificate management: Integrating with existing PKI infrastructures
- Regulatory compliance: Meeting industry-specific security requirements
- Backward compatibility: Supporting legacy systems with limited cryptographic capabilities

Feng et al. addressed these challenges through a layered architectural approach that provided clean abstraction boundaries between the cryptographic implementation and the application layer, allowing gradual migration from legacy systems [55].

5.7.6 Comparative Analysis of Hybrid Implementations

A comparative analysis of the hybrid cryptographic implementations examined in this section reveals several patterns and insights that can inform future designs.

5.7.6.1 Performance Comparison

Table 5.5 summarizes the performance characteristics of the examined hybrid implementations across key metrics.

RSA–ECC regularly exhibits better performance across all criteria, showcasing the best throughput and minimal memory utilization. This indicates that the integration of RSA and ECC offers an ideal equilibrium between security and performance for numerous applications [13].

Table 5.5 Performance Comparison of Hybrid Cryptographic Implementations

Implementation	*Encryption throughput (%)*	*Decryption throughput (%)*	*Memory usage (KB)*	*Key size (symmetric/ asymmetric) (bits)*
AES–RSA [55]	29.83	26.54	6.908	128/2048
AES–ECC [12]	19.86	14.76	8.916	128/256
RSA–ECC [13]	63.10	58.38	2.372	N/A/2048-256
Blowfish–RSA [54]	35.26	45.81	6.372	448/2048

5.7.6.2 Security Analysis

Security analysis reveals that hybrid implementations generally provide stronger security guarantees than single-algorithm approaches. Key findings include the following:

1. Resistance to quantum attacks: Hybrid systems combining multiple asymmetric algorithms provide some defense against quantum computing threats by requiring adversaries to break multiple cryptographic problems.
2. Defense in depth: Layered encryption provides protection even if one algorithm is compromised.
3. Algorithm diversity: Using different algorithms reduces the impact of algorithm-specific vulnerabilities.

Alatawi's security research indicated that hybrid methodologies can efficiently address recognized vulnerabilities in specific algorithms while preserving satisfactory performance [14].

5.7.6.3 Application-Specific Considerations

The comparison research indicates that ideal hybrid implementations are significantly application-specific, with various domains emphasizing distinct elements of the security-performance trade-off.

- Cloud computing: Emphasizes throughput for bulk data encryption while maintaining strong key security
- IoT security: Prioritizes minimal resource consumption with adequate security guarantees
- Financial systems: Demands the highest security levels with acceptable performance
- Mobile applications: Requires battery-efficient implementations with strong user authentication

This application specificity underscores the necessity of customizing hybrid cryptographic systems to meet the distinct requirements of each domain rather than employing a universal approach [5].

5.7.6.4 Future Directions in Hybrid Cryptographic Implementations

The examination of current hybrid cryptographic implementations points to several promising directions for future research and development:

1. Quantum-resistant hybrids: Integrating post-quantum algorithms with classical cryptography to provide transitional security
2. Adaptive cryptography: Dynamically selecting algorithms and parameters based on security requirements and resource availability

3. Homomorphic extensions: Combining partially homomorphic properties with hybrid encryption for secure computation
4. Hardware-software co-design: Developing specialized hardware accelerators for hybrid cryptographic operations
5. Formal verification: Applying formal methods to verify the security properties of hybrid implementations

As security requirements grow and new threats arise, hybrid cryptographic implementations will likely assume a more significant role in delivering robust and flexible security solutions across many application domains [7].

5.8 CONCLUSION AND FUTURE SCOPE

This chapter provides an in-depth look at hybrid cryptographic systems, emphasizing their essential function within contemporary security frameworks. Hybrid cryptography has advanced considerably from the basic integration of symmetric and asymmetric methods to the development of complex security frameworks that tackle the diverse challenges present in modern digital environments. This chapter illustrates how hybrid cryptosystems utilize the complementary strengths of various cryptographic paradigms. They employ symmetric encryption for efficient bulk data processing while relying on the robust key management capabilities of asymmetric encryption for secure key exchange and distribution [3, 4].

The performance assessments in Sections 5.3 and 5.7 demonstrate that well-structured hybrid techniques consistently surpass standalone cryptographic algorithms regarding the security-efficiency trade-off. The RSA–ECC combination has exhibited enhanced performance metrics characterized by increased throughput and reduced memory consumption, rendering it particularly appropriate for resource-limited settings [13]. The incorporation of diverse transformation methods and several encryption layers in hybrid systems has markedly improved resilience against cryptanalytic assaults, including those employing quantum computing technologies [54].

Case examples in Section 5.7 demonstrate the versatility and adaptability of hybrid cryptosystems across several application areas, including cloud computing, IoT networks, financial transactions, and healthcare systems. These implementations illustrate that hybrid methodologies can be adeptly customized to satisfy domain-specific needs while preserving strong security assurances [5, 8].

5.8.1 Challenges and Limitations

Hybrid cryptographic systems, despite their benefits, encounter numerous substantial problems that must be resolved for broader implementation and efficacy. The complexity of key management continues to be a significant issue, as hybrid systems necessitate the management of both symmetric and asymmetric keys, which may have varying lifespan requirements and security procedures [54]. This complexity heightens the likelihood of implementation errors and may expose vulnerabilities that could jeopardize the overall security architecture.

Resource limitations are a significant barrier, especially in IoT and mobile contexts, where processor capacity, memory, and energy supply are restricted. Although lightweight hybrid cryptographic systems like LSA have been created to mitigate these limitations, they frequently entail security-efficiency trade-offs that could diminish resilience against specific cryptanalytic assaults [12]. The additional computational burden from several encryption levels can considerably affect processing delay and energy usage in battery-operated devices [14].

Integrating with legacy systems poses further hurdles, as numerous firms have significant investments in their current cryptographic infrastructure. Maintaining backward compatibility while adopting more secure hybrid methodologies necessitates meticulous architecture design and frequently entails intricate transitional processes [55]. Integration problems may impede the adoption of advanced cryptographic technologies, rendering systems susceptible to growing attacks.

5.8.2 Future Research Directions

As cryptographic threats advance, numerous interesting possibilities for study are emerging within the domain of hybrid cryptography. Post-quantum hybrid cryptography is a pivotal domain for future advancement, propelled by the imminent risk that quantum computing poses to conventional asymmetric encryption techniques [48]. The amalgamation of quantum-resistant algorithms with classical cryptography will be crucial for ensuring transitional security during the shift to entirely quantum-resistant systems. NIST's current standardization initiatives are anticipated to significantly influence the future framework of post-quantum hybrid cryptography [56].

Adaptive and context-aware cryptographic frameworks signify a promising avenue, emphasizing systems capable of dynamically selecting suitable algorithms and settings according to security requirements, threat landscape, and resource availability. Hybrid systems employing adaptive key management strategies alongside time-constrained access control mechanisms have shown considerable enhancements in overall security via the dynamic production of tailored encryption keys [9]. Subsequent investigations in this domain may yield more advanced self-regulating cryptographic systems that enhance the security-performance balance in real time [7].

The combination of artificial intelligence and hybrid cryptography presents significant opportunities for improving security mechanisms. Hybrid encryption methods utilizing machine learning, particularly those that integrate genetic algorithms for dynamic key generation, demonstrate potential in enhancing encryption randomness and bolstering resistance against cryptanalytic attacks [10]. Future research may investigate advanced AI methodologies for forecasting and mitigating new cryptographic threats, enhancing algorithm selection processes, and identifying potential vulnerabilities in hybrid implementations.

5.8.3 Emerging Applications

The advancement of distributed systems and blockchain technology introduces both opportunities and challenges for hybrid cryptographic applications. The incorporation of hybrid signatures, such as BLS combined with ECDSA, within blockchain platforms like Ethereum 2.0 has shown enhancements in both scalability and security [52]. Future research may investigate advanced hybrid key combiners that incorporate pre-quantum, post-quantum, and QKD algorithms to ensure comprehensive security guarantees [35].

Cloud and edge computing environments are expected to persist in fostering innovation in hybrid cryptographic systems, driven by their varied security requirements and resource limitations. Automated security frameworks for cloud storage utilize hybrid cryptography to autonomously select suitable encryption algorithms, taking into account data sensitivity and performance requirements [7]. The combination of FHE techniques with hybrid encryption facilitates secure computation on encrypted data without the need for decryption, thereby creating new opportunities for privacy-preserving cloud services [37].

The implementation of cryptographic measures in IoT and smart city infrastructures presents significant challenges, primarily due to their heterogeneous characteristics and

constraints on resources. Hybrid cryptographic systems that are lightweight and specifically tailored for IoT devices have shown notable enhancements in execution time and memory utilization when compared to traditional algorithms [12]. Future applications may encompass advanced hybrid encryption models designed for secure authentication protocols within IoT networks. This approach aims to tackle the essential challenge of secure device validation while maintaining computational efficiency [5].

In conclusion, hybrid cryptographic systems signify a dynamic and evolving domain that persistently adapts to emerging security issues and technical breakthroughs. By adeptly integrating the advantages of several cryptographic methodologies, these systems present a viable avenue for enhanced, efficient, and versatile security solutions in the more intricate digital environment. The combination of post-quantum algorithms, adaptive frameworks, and AI-enhanced mechanisms will undoubtedly influence the future generation of hybrid cryptographic systems, assuring their relevance in protecting digital assets and communications amid rapidly expanding cyber threats.

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Chapter 6

Secure Image Encryption Based on a Novel 2D Chaotic Map and Four-Phase Scrambling

Shalini Gupta, Nitish, and Anand Nayyar

6.1 INTRODUCTION

With the rapid advancement of digital technology, the use of image data has surged across numerous sectors such as healthcare, finance, communication and surveillance. Every day, vast amounts of images are captured, transmitted through networks, stored on cloud servers and accessed by a wide range of connected devices. However, this increasing reliance on digital images raises significant concerns regarding privacy, data integrity and unauthorized access. As a result, ensuring the secure transmission and storage of image data has become a critical challenge in the field of information security.

To address these concerns, image encryption has emerged as a vital technique for protecting sensitive visual information. Unlike text-based data, digital images possess unique characteristics such as high redundancy, strong correlations between neighboring pixels and large file sizes. These properties make traditional encryption algorithms like the Advanced Encryption Standard or Data Encryption Standard less effective when applied directly to image data. Consequently, researchers have focused on developing encryption methods specifically tailored to images which offer both enhanced efficiency and robust security [1, 2].

The growing interest in image encryption techniques stems from the need to protect the confidentiality of visual data in the digital world. As images are increasingly used in critical applications, they become vulnerable to various security threats such as unauthorized access, tampering or interception during transmission [3]. To protect image data, numerous techniques had been proposed including the use of chaotic systems [4, 5, 6], quantum computing [7], DNA sequence operations [8, 9] and elliptic curves [10]. These methods offer unique solutions to the challenges of image encryption, each providing specific advantages depending on the application context.

Among these techniques, chaotic systems have gained significant attention due to their inherent unpredictability, sensitivity to initial conditions and pseudo-random behavior [11, 12]. These characteristics make chaotic maps ideal for cryptographic applications. Chaotic systems have been integrated into many encryption schemes, particularly in performing pixel permutation, diffusion and substitution operations. The selection between one-dimensional and high-dimensional chaotic maps for image encryption involves a trade-off between algorithmic simplicity and computational overhead. Zhang et al. [13] introduced an innovative image encryption method utilizing a polynomial chaotic map, leading to the construction of a novel three-dimensional (3D) chaotic system characterized by intricate dynamics and heightened computational demands. Similarly, Liang et al. [14] developed a medical image encryption algorithm that integrates a new five-dimensional three-leaf chaotic model with genetic operations. On the other hand, Khairullah et al. [15] focused on enhancing encryption efficiency by proposing optimized one-dimensional chaotic maps, including improved versions of the logistic and quadratic maps. These approaches also included a key management

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mechanism to reduce the number of iterations of the chaotic functions, thereby accelerating the encryption process.

Emerging research has spotlighted fractional-order hyperchaotic systems as promising candidates for secure image encryption [16]. For instance, Gao et al. [17] proposed a method that encrypts multiple images using a 3D cube combined with a hyperchaotic map. Meanwhile, Hosny et al. [18–20] explored hybrid encryption frameworks integrating Gegenbauer moments, block scrambling, two-dimensional coupled chaotic maps (2D CCM) and fractional-order hyperchaotic dynamics. Another notable development by Li and Chen [21] introduced an encryption scheme based on a six-dimensional hyperchaotic system enhanced with DNA encoding, enabling greater complexity and security. These fractional-order systems typically rely on fractional differential equations and combine traditional and non-traditional cryptographic mechanisms, resulting in robust, nonlinear behavior with strong resistance to attacks.

Yu et al. [22] proposed a sophisticated encryption technique that employed a memristor-coupled resonant tunneling diode cellular neural network (RTD-MCNN), optimized for secure forensic image protection. Exhibiting dynamic chaotic behavior such as multistability and initial condition sensitivity, the system leverages two positive Lyapunov exponents for improved security. Building on this, Yu et al. [23] introduced a four-dimensional fractional-order memristive Hopfield neural network (4D-FOMHNN), which demonstrated transient chaos and coexisting attractors, validated both numerically and via FPGA implementations. The model capitalizes on the memory-enhancing properties of fractional calculus and memristors, offering a viable hardware-ready solution for secure encryption applications.

Ma et al. [24] developed an image encryption framework that integrates Chen's hyperchaotic system with the Tabu Search optimization technique to enhance scrambling via sub-block operations. In this scheme, encryption keys are derived from the hash of the input image to ensure sensitivity to plaintext. Simulation experiments reveal the scheme's strong defense against both differential and ciphertext-only attacks. Deng et al. [25] proposed a memristive neural model for image encryption that employed a piecewise nonlinear memristor to generate multi-wing chaotic attractors, such as the double-wing butterfly structure. Verified through FPGA implementations, the model's adjustable chaotic dynamics offer enhanced security potential.

Addressing performance inefficiencies in earlier chaos-based schemes, Feng et al. [26] introduced the MIEA-PRHM algorithm, which employed dual hyperchaotic maps and multi-channel processing. It applies unique transformations on 4-bit image blocks, improving key space size, randomness and encryption efficiency. In a subsequent study, Feng et al. [27] presented the IEASP algorithm, which merges a robust two-dimensional SQPM hyperchaotic map with pixel fusion and vector-level filtering techniques. These innovations improve entropy and the number of pixel change rate (NPCR) while delivering faster encryption rates, surpassing existing methods in performance metrics.

Recent works emphasize the importance of evaluating both chaotic map structure and encryption process design in developing secure image encryption systems [28]. A detailed cryptanalysis of an image encryption method based on a 2D logistic-adjusted sine map revealed significant vulnerabilities, including susceptibility to chosen-plaintext attacks (CPAs), where attackers were able to recover original images without access to encryption keys [29]. This highlights the necessity of robust architectural design alongside chaotic dynamics.

Another analysis examined a scheme that combined Feistel network architecture with dynamic DNA encoding [30]. The findings revealed flaws in key generation and encryption workflows, rendering the system vulnerable to CPAs. This stresses the critical need for comprehensive cryptanalysis during the design stage of encryption systems to proactively identify and address weaknesses.

Furthermore, Wen and Lin [7] assessed an encryption method incorporating quantum chaotic maps and DNA coding, identifying weaknesses in key management and vulnerability to CPAs. This line of work was expanded by Wen and Lin [9], where additional weaknesses were found in DNA and multi-chaotic map-based encryption algorithms via statistical and differential attacks. Another study by Wen et al. [31] analyzed a chaotic Hill cipher-based scheme, exposing flaws that enabled plaintext recovery through algebraic attacks. Notably, You et al. [32] applied a convolutional neural network to break chaotic image encryption, showing that machine learning can effectively decrypt such encrypted data.

Collectively, these cryptanalytic efforts suggest that for chaotic encryption systems to be secure, future schemes must better integrate chaos generation with encryption operations. Strengthening the linkage between permutation and diffusion phases, while enhancing the dynamism and unpredictability of chaotic sequences, is essential for resisting known-plaintext and chosen-ciphertext attacks.

However, many of the existing algorithms based on chaotic maps still face limitations such as low complexity, small key spaces or poor resistance to differential attacks. These limitations necessitate the development of more robust and efficient image encryption algorithms.

In response to these challenges, the present chapter proposes Novel Image Encryption Algorithm (NIEA) based on a newly designed 2D CCM. This approach generates multiple independent chaotic sequences that drive a four-stage encryption process: pixel permutation, DNA-inspired encoding, pixel diffusion and bit-level reversion. Each stage in this encryption framework contributes to the overall confusion and diffusion of the image data, significantly enhancing the resistance of the encrypted image to statistical and differential attacks [31, 32, 33].

The main contributions of this chapter include the following:

- The design of a new 2D CCM that produces highly sensitive and unpredictable chaotic sequences suitable for cryptographic applications.
- A lightweight four-stage encryption framework that combines pixel permutation, DNA-based encoding, diffusion and bit-level transformation.
- A simple, efficient and effective encryption scheme for color images that can also be extended to real-time applications.

Organization of the Chapter

The rest of the chapter is organized as follows. Section 6.2 illustrates preliminaries. Section 6.3 describes the mathematical formulation of the proposed 2D CCM and its dynamic properties. Section 6.4 presents the step-by-step image encryption process, including chaotic sequence generation, pixel transformation and bit-level operations. Section 6.5 provides experimental results and performance evaluations including statistical analysis and comparison with other methods. And, finally, Section 6.6 concludes the chapter with future scope.

6.2 PRELIMINARIES

6.2.1 DNA-Encoding Rules

DNA, in its biological context, is a molecule that stores genetic information using four basic nucleotides: adenine (A), thymine (T), cytosine (C) and guanine (G). In digital encoding, each of these nucleotides can be represented using 2-bit binary values—for instance, A as 00, C as 01, G as 10 and T as 11. Using this binary representation, 24 unique encoding rule sets can be derived, though these are typically condensed to eight distinct schemes based on Watson–Crick complementary **base pairing rules**.

Table 6.1 DNA-Encoding Rules

Rule	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
00 (A)	A	A	C	G	C	G	T	T
01 (C)	C	G	A	A	T	T	C	G
10 (G)	G	C	T	T	A	A	G	C
11 (T)	T	T	G	C	G	C	A	A

In image encryption, each pixel value (ranging from 0 to 255) is first converted to an 8-bit binary string. This string is then interpreted as a four-nucleotide DNA sequence using one of the selected DNA rules (see Table 6.1). Through such transformations, the DNA-encoding layer introduces a biologically inspired, yet computationally effective means to manipulate pixel data. This not only boosts entropy and reduces correlation among image pixels but also adds another layer of confusion, thereby improving the overall robustness of the encryption scheme.

6.3 NOVEL 2D-COUPLED CHAOTIC MAP

Chaotic maps are a subclass of nonlinear dynamical systems known for their ability to generate complex, aperiodic and highly sensitive sequences when subjected to specific initial conditions and parameter configurations. These intrinsic properties make them particularly well-suited for cryptographic applications, especially in the domain of image encryption, where high unpredictability and randomness are essential.

In this section, we present a newly constructed 2D CCM that serves as the foundational component of the NIEA framework. This map has been carefully formulated to exhibit the essential characteristics of a secure chaotic system sensitivity, ergodicity, unpredictability and high structural complexity, while maintaining computational efficiency.

6.3.1 Mathematical Definition

The 2D CCM is defined by the following iterative system:

$$x_{n+1} = \left(a \cdot sin\left(\pi \cdot \left(y_n + 3\right) \cdot x_n\right) + \frac{1}{y_n} \right) \; mod \; 1$$

$$y_{n+1} = \left(b \cdot \left(1 - y_n\right) + \frac{1}{x_n} \right) mod 1$$

Here, x_n and y_n denote the state variables at the nth iteration and a and b are control parameters that determine the strength and structure of the chaotic behavior. The modulo 1 operation is applied to constrain the outputs within the interval $(0,1)$, ensuring bounded and normalized sequences. This system is nonlinear due to its combination of trigonometric and reciprocal functions, which interact multiplicatively with one another. These nonlinear interactions greatly increase the complexity of the generated sequences and reinforce the system's cryptographic utility. To initiate the chaotic sequence, the system requires four critical values: two initial conditions: x_0 and y_0; two control parameters: a and b. These four parameters form the core secret key for the chaotic system. Even an infinitesimal variation in any of these values can lead to drastically different sequence outputs—an essential property known as sensitivity

to initial conditions. This is a cornerstone of chaos-based encryption systems, ensuring that unauthorized decryption attempts using slightly incorrect parameters will fail completely.

The proposed 2D CCM is deliberately designed to satisfy several critical properties required for secure cryptographic systems:

- *Sensitivity to Initial Conditions*
  Tiny perturbations in x_0 or y_0 as small as 10^{-14} can result in wholly divergent sequence trajectories. This high sensitivity ensures that even a slight deviation in the key leads to a completely different ciphertext, providing robustness against brute-force and differential attacks.
- *Ergodicity*
  Over a sufficiently large number of iterations, the state variables x_n and y_n span the entire phase space within the unit interval. This statistical property ensures the uniform coverage of values and contributes to the high randomness of the output.
- *Unpredictability*
  Due to the use of nonlinear sine and inverse terms, the system exhibits complex and irregular behavior. The sequence lacks any discernible patterns or periodicity, rendering it unpredictable even with partial knowledge of the system's parameters.
- *High-Dimensional Interaction and Complexity*
  Unlike traditional one-dimensional maps (e.g., Logistic or Tent maps), the proposed 2D CCM involves the interdependent evolution of two state variables. This interdimensional coupling increases the overall system entropy and sequence unpredictability, which in turn enhances the cryptographic strength of the generated chaotic outputs.

6.3.2 Dynamic Analysis of the 2D CCM

To comprehensively assess the chaotic characteristics and cryptographic robustness of the 2D CCM, a series of established dynamic analysis techniques was employed. These evaluations include the visualization of phase space trajectories, computation of approximate entropy, assessment of sequence uniformity and examination of the system's key sensitivity. Additionally, the output sequences were subjected to the TestU01 randomness test suite, a rigorous benchmark for pseudo-random number generators. Collectively, these analyses demonstrate that the 2D CCM generates highly unpredictable, statistically independent and secure chaotic sequences, affirming its suitability for use in advanced image encryption frameworks.

6.3.3 Phase Space Analysis

The phase space trajectory offers a graphical representation of the dynamic evolution of the system's state variables x_n and y_n over time. In a robust chaotic system, the phase diagram should exhibit a dense and irregular scatter, devoid of any visible periodicity or repetitive patterns, indicating high sensitivity and nonlinearity. To visualize the phase space of the 2D CCM, the following configuration is used:

- Control parameters are fixed at $a = 3.78,\ b = 0.78$.
- Initial conditions are set to $x_0 = 0.621,\ y_0 = 0.731$.
- The system is iterated for 1000 steps, and the trajectory is plotted in the (x_n, y_n) plane.

The resulting plot reveals a scattered and non-repeating distribution of points, affirming the map's chaotic nature. Moreover, this irregular pattern remains consistent across a broad range of parameter values $a \in (1,10]$ and $b \in (1,10]$, highlighting the robustness and generality of

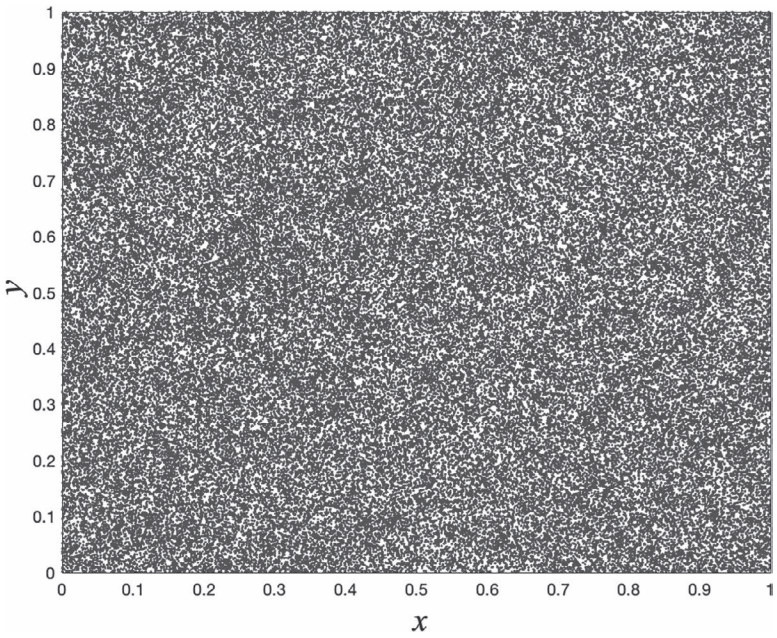


Figure 6.1 Phase space of 2D CCM.

the chaotic behavior. Representative phase plot is illustrated in Figure 6.1, showcasing the ergodic coverage and nonlinear coupling of the 2D CCM.

6.3.4 Bifurcation Behavior

To further verify the chaotic dynamics of the 2D CCM, a bifurcation diagram is constructed by systematically varying one of the control parameters while holding the other constant. In this analysis, the parameter a is fixed at 3.78, and the parameter b is varied continuously within the range $b \in (0,4]$. As b increases, the system exhibits complex and irregular patterns indicating the onset of chaos. The resulting bifurcation diagram, shown in Figure 6.2, plots the long-term values of x_n against corresponding values of b. The dense structure observed in the whole region of the plot confirms the presence of chaotic regimes, where the system becomes highly sensitive to both control parameters and initial conditions. This analysis highlights the parameter dependency and dynamic richness of the 2D CCM, reinforcing its suitability as a core engine for secure and unpredictable image encryption.

6.3.5 Approximate Entropy (ApEn)

Approximate entropy (ApEn) is a statistical measure used to quantify the unpredictability and complexity within a time series. In the context of encryption, a higher ApEn value reflects a greater degree of randomness, which is essential for ensuring the security and robustness of cryptographic systems. For a given sequence $\{x_1, x_2, ..., x_N\}$, , ApEn is computed using the following expression:

$$\mathrm{ApEn}(m, r, N) = \phi^m(r) - \phi^{m+1}(r)$$

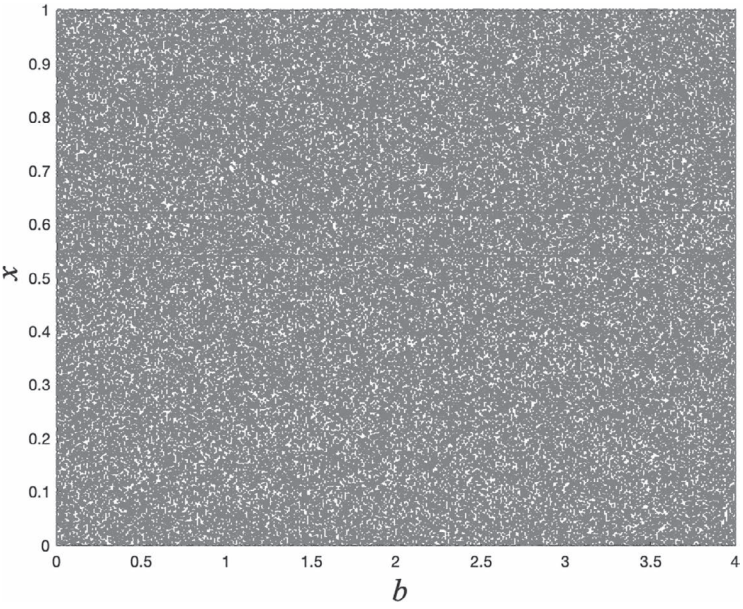


Figure 6.2 Bifurcation diagram of 2D CCM.

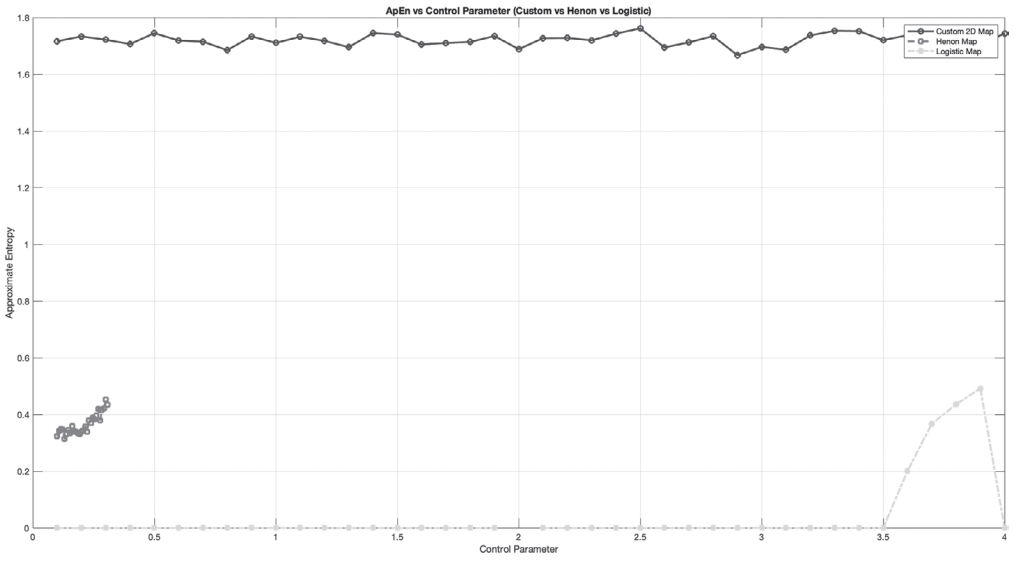


Figure 6.3 Approximate entropy curve for logistic, Henon map and 2D CCM.

where m is the embedding dimension (e.g., 2), r is the tolerance typically defined as a fraction of the standard deviation ($r = 0.2 \times$ standard deviation), and $\phi^m(r)$ represents the average similarity between sequences of length m within the dataset. A value of ApEn approaching 1 signifies high sequence randomness and low predictability—both desirable characteristics for cryptographic applications. In our analysis, the ApEn values of the 2D CCM were compared against those generated by traditional chaotic maps, such as the Logistic Map and the Henon Map. The comparison, visualized in Figure 6.3, demonstrates that the

2D CCM consistently yields higher ApEn values, indicating its superior capability to produce complex and unpredictable sequences suitable for secure encryption.

6.3.6 Sequence Uniformity

The uniformity of a chaotic sequence's distribution is a key indicator of its randomness quality and by extension its effectiveness in cryptographic applications. A uniform distribution ensures that all possible values occur with approximately equal probability, thereby minimizing statistical predictability and strengthening resistance against analytical attacks. In contrast, sequences with non-uniform distributions exhibit bias, making them more vulnerable to statistical exploitation. Figure 6.4 illustrates the sequence uniformity comparison across the traditional logistic map. As observed, the logistic map produces a distribution with noticeable irregularities and clustering, reflecting its limited randomness and reduced cryptographic strength. In contrast, the output from the 2D CCM exhibits a uniform distribution, confirming its ability to generate statistically balanced sequences. This uniformity underlines the 2D CCM's capability to produce secure, unpredictable outputs, making it a strong foundation for reliable encryption mechanisms.

6.3.7 Key Sensitivity Analysis

One of the fundamental requirements for a secure chaotic encryption system is extreme sensitivity to initial conditions and key parameters. This characteristic ensures that even minuscule changes in the input parameters result in completely divergent output sequences, thereby enhancing resistance to brute-force and differential attacks. To evaluate this property in the 2D CCM, a controlled experiment was conducted. The system was first initialized with $x_0 = 0.621$, and a chaotic sequence was generated over 200 iterations. In a second run, a slight perturbation was introduced, setting $x_0' = 0.62100000000001$, while keeping all other parameters constant. The resulting sequences were then compared to assess divergence. As shown in Figure 6.5, the sequences generated by the 2D CCM begin to deviate noticeably within the first six iterations, highlighting the system's high key sensitivity. In contrast, the logistic map used here as a baseline exhibits discernible divergence only after approximately 60 iterations under the same perturbation scale. This comparison underscores the superior sensitivity of the 2D CCM, which is a desirable property in encryption systems, as it ensures that even the smallest variation in the secret key leads to significantly different ciphertext outputs, thereby enhancing overall cryptographic security.

6.3.8 Lyapunov Exponent

To further establish the chaotic characteristics of the 2D CCM, the Lyapunov exponent was computed. The Lyapunov exponent is a key metric in nonlinear dynamics used to assess a system's sensitivity to initial conditions. A positive Lyapunov exponent indicates that nearby trajectories diverge exponentially over time, a defining feature of chaotic behavior. For a two-dimensional system, there are generally two Lyapunov exponents. If at least one is positive and the combined sum does not diverge excessively, the system is considered to exhibit chaotic dynamics. In this study, the Lyapunov exponents were calculated across a broad range of the control parameter $b \in (0,10]$, while keeping other parameters fixed. The results, illustrated in Figure 6.6, demonstrate that the system consistently maintains positive Lyapunov values throughout the examined parameter space. This confirms that the 2D CCM maintains chaotic stability across various configurations, further validating its effectiveness as a robust and sensitive chaotic generator. Such behavior is highly desirable in encryption applications,

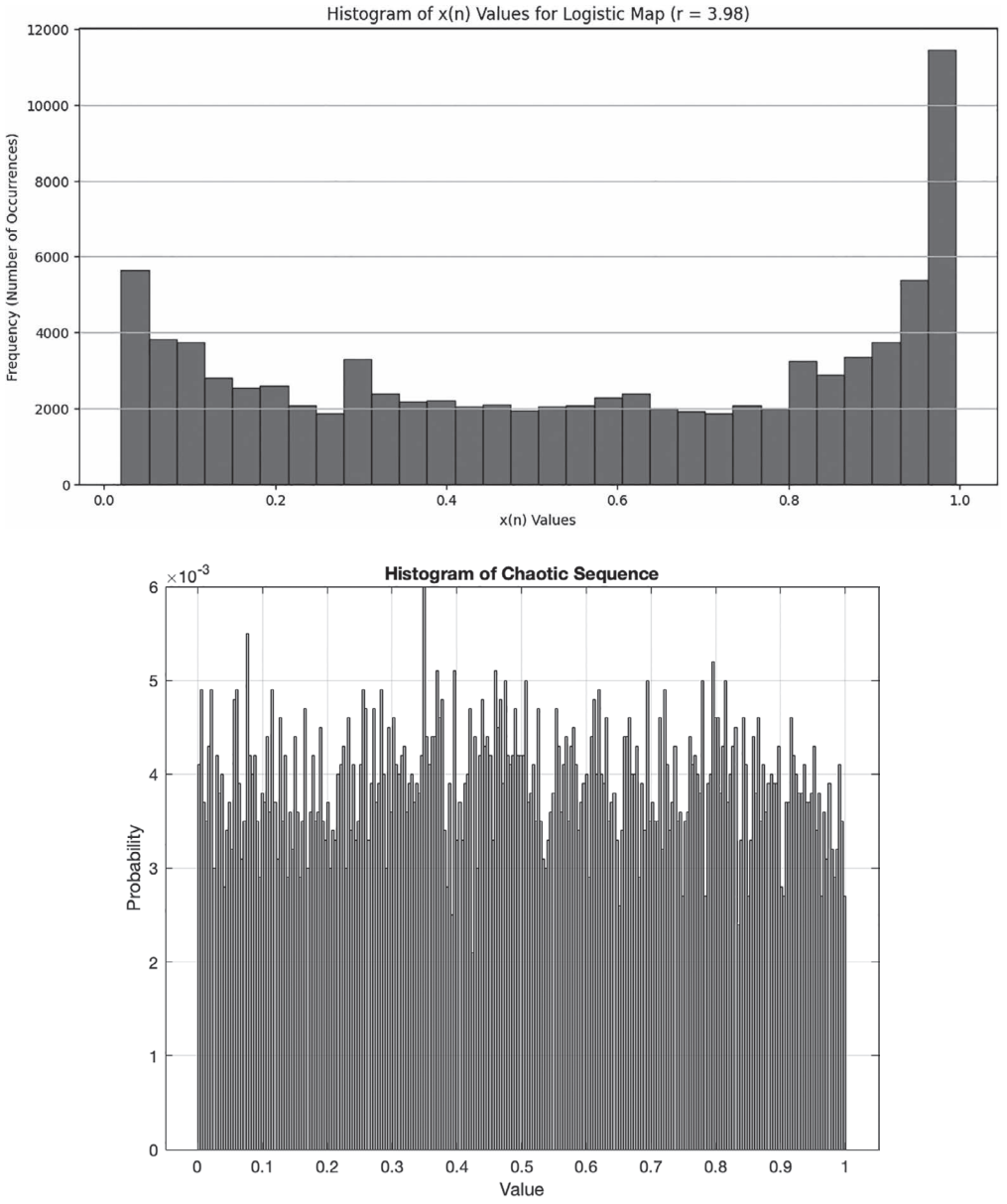


Figure 6.4 Sequence uniformity plot for the logistic map and the 2D CCM, respectively.

as it ensures the production of complex, unpredictable and highly sensitive sequences, making the 2D CCM well-suited for modern cryptographic systems.

6.3.9 Randomness Testing

TestU01 is widely recognized as one of the most comprehensive and rigorous frameworks for assessing the randomness quality of pseudo-random number generators (PRNGs) [34]. The suite comprises three progressively stringent test batteries: Small Crush, Crush and Big Crush, each designed to evaluate increasingly subtle statistical anomalies in random sequences. In

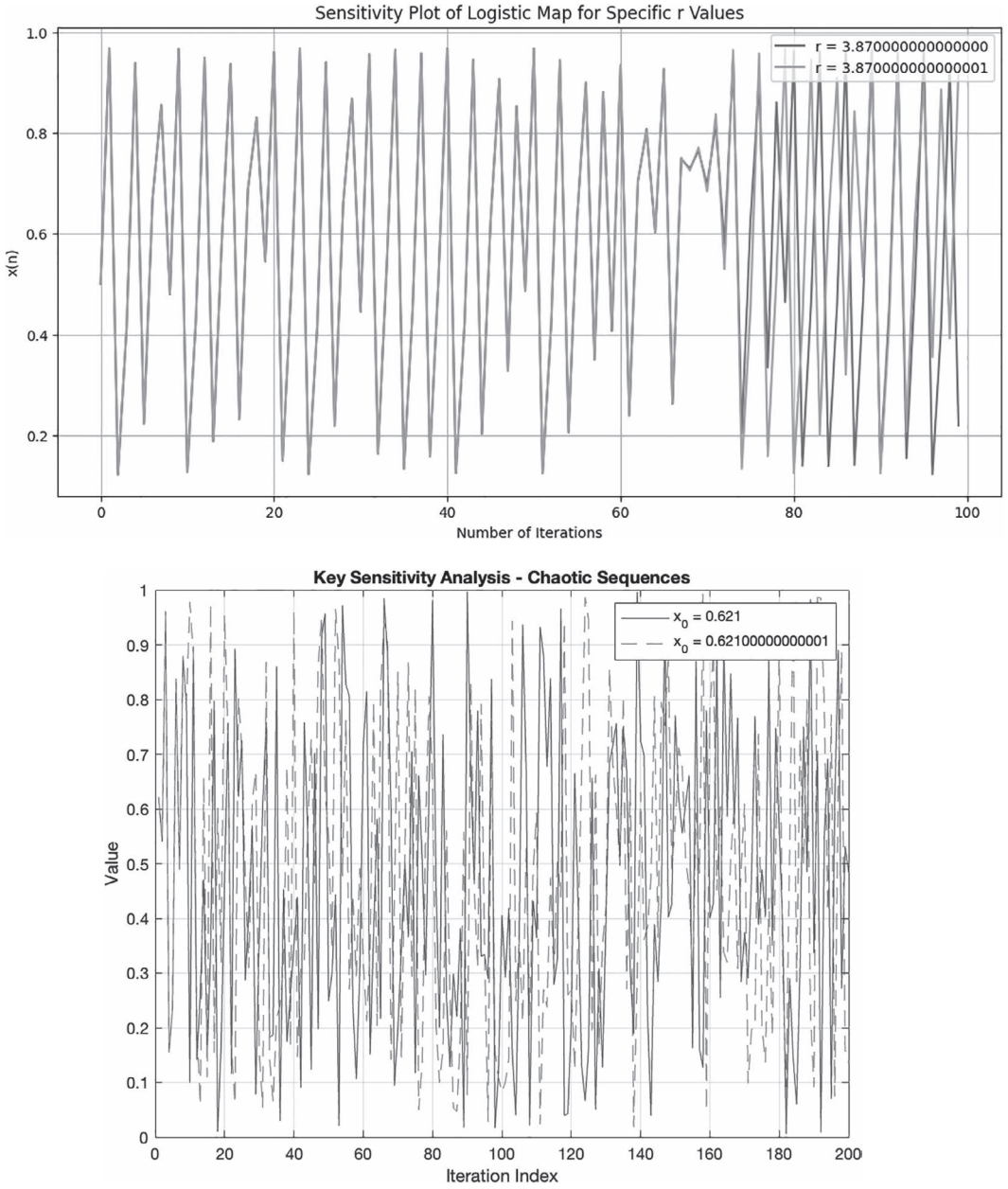


Figure 6.5 Map sensitivity plot for the logistic map and 2D CCM.

this study, we employed TestU01 to evaluate the statistical integrity of sequences generated by the 2D CCM. The analysis was conducted using output sequences scaled by gain values of 10^{13}, 10^{14}, 10^{15} and 10^{16} respectively. These gain values were selected to assess the consistency of randomness across different magnitude scales. The number of failed tests under each configuration is summarized in Table 6.2. The results indicate that the sequences generated by the 2D CCM pass the vast majority of statistical tests, with only a negligible number of failures observed, primarily under the most stringent conditions. This outcome demonstrates

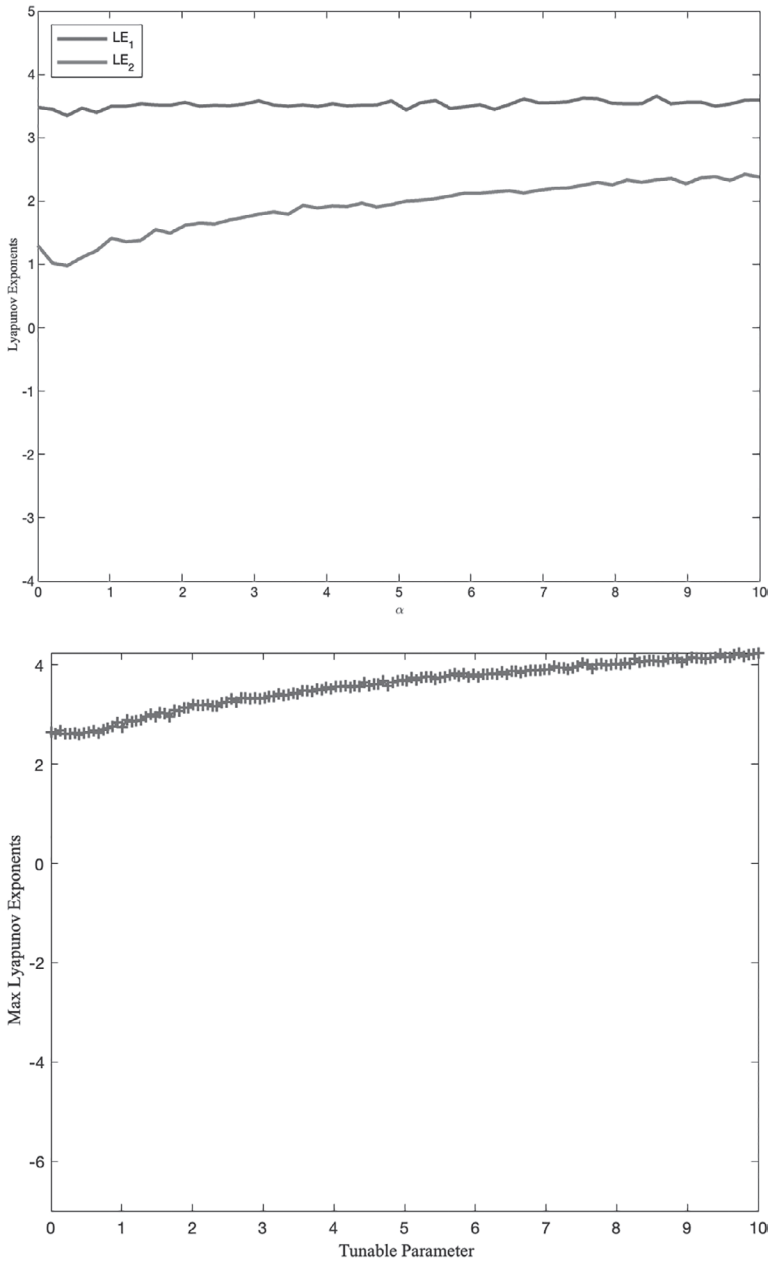


Figure 6.6 Lyapunov exponent and maximum Lyapunov exponent for different tunable parameters.

that the 2D CCM exhibits strong statistical randomness, validating its reliability and suitability for cryptographic applications that demand high-quality random number generation.

6.4 PROPOSED IMAGE ENCRYPTION ALGORITHM

This section introduces the NIEA, which integrates a newly designed 2D CCM with a layered transformation structure to achieve a secure and efficient image encryption process. The core

Table 6.2 Number of Failed Statistical Tests in the TestU01 Suite for Different Gain Values

Gain		10^{13}	10^{14}	10^{15}	10^{16}
2D CCM (x)	Small Crush	0	0	0	0
	Crush	0	0	0	0
	Big Crush	2	1	1	1
2D CCM (y)	Small Crush	0	0	0	0
	Crush	0	0	0	0
	Big Crush	3	2	1	0

innovation in NIEA lies in its ability to generate multiple high-entropy chaotic sequences, each governed by unique initial conditions and control parameters. These sequences independently drive the encryption pipeline, resulting in strong sensitivity to initial conditions, enhanced unpredictability and improved resistance to various cryptanalytic attacks. The encryption process is composed of four sequential stages, each designed to progressively increase the randomness and security of the image. It begins with pixel permutation, which rearranges the spatial distribution of image pixels, effectively breaking the natural correlation found in unprocessed images. This is followed by DNA-based encoding, where pixel values are mapped to DNA sequences and manipulated using biologically inspired rules to introduce nonlinearity and further obfuscate pixel relationships. The third stage, pixel diffusion, ensures that minor changes in the input propagate throughout the entire image, amplifying the avalanche effect critical for strong encryption. Finally, the bit-level reversion step manipulates the binary representation of pixel values to eliminate any remaining patterns, increasing both complexity and entropy in the ciphertext. Let $I \in R^{M\times N\times 3}$ represent the input RGB color image of dimensions $M\times N$. The NIEA transforms the input image I into an encrypted image $E \in R^{M\times N\times 3}$ by applying the four stages in sequence as follows:

6.4.1 Key Generation

To guarantee both security and uniqueness in the encryption process, the initial conditions and control parameters for the 2D CCM are not statically defined but are dynamically derived from the input image itself. This is achieved through a hash-driven key generation mechanism, which ensures that even the slightest change in the image content produces an entirely different encryption key. Specifically, the widely used and cryptographically secure SHA-256 hash function is applied to the original image to generate a 256-bit digest. This hash output is then parsed and transformed into the required chaotic system parameters, including initial values and control variables, enabling the generation of image-dependent, high-entropy chaotic sequences. This strategy strengthens key sensitivity, prevents reuse of encryption keys and resists CPAs and known-plaintext attacks by tightly coupling the key space to the unique content of the image.

Steps:

1. Given an image I, an external secret key k and the bitwise XOR operation, the SHA-256 hash is calculated as

$$\mathrm{SHA}_{256}(I \oplus k) = x.$$

2. Divide the x into 32 segments with an 8-bit length:

$$x_{256} = x_1 x_2 x_3 \cdots x_{32}.$$

3. Using the segments x_i, the initial values $v_i^{'}$ and control parameters a_i and b_i for the chaotic system are computed as follows:

$$v_1^{'} = \left(\left(x_1 \oplus x_2 \oplus x_3 \oplus x_4 \right) / \sum_{i=1}^{8} x_i \right) mod\ 1$$

$$v_2^{'} = \left(\left(x_5 \oplus x_6 \oplus x_7 \oplus x_8 \right) / \sum_{i=1}^{16} x_i \right) mod\ 1$$

$$v_3^{'} = \left(\left(x_9 \oplus x_{10} \oplus x_{11} \oplus x_{12} \right) / \sum_{i=1}^{24} x_i \right) mod\ 1$$

$$v_4^{'} = \left(\left(x_{13} \oplus x_{14} \oplus x_{15} \oplus x_{16} \right) / \sum_{i=1}^{32} x_i \right) mod\ 1$$

$$v_5^{'} = \left(\left(x_{17} \oplus x_{18} \oplus x_{19} \oplus x_{20} \right) / \sum_{i=1}^{32} x_i \right) mod\ 1$$

$$a_1 = mod((x_{21} \oplus x_{22} \oplus x_{23} \oplus x_{24}) \times 10^{-12}, 10) + 1$$

$$a_2 = mod((x_{25} \oplus x_{26} \oplus x_{27} \oplus x_{28}) \times 10^{-12}, 10) + 1$$

$$a_3 = mod((x_{29} \oplus x_{30} \oplus x_{31} \oplus x_{32}) \times 10^{-12}, 10) + 1$$

$$a_4 = mod((x_8 \oplus x_{16} \oplus x_{24} \oplus x_{32}) \times 10^{-12}, 10) + 1.$$

$$b_1 = mod((x_1 \oplus x_3 \oplus x_5 \oplus x_7) \times 10^{-12}, 10) + 1$$

$$b_2 = mod((x_2 \oplus x_4 \oplus x_6 \oplus x_8) \times 10^{-12}, 10) + 1$$

$$b_3 = mod((x_9 \oplus x_{11} \oplus x_{13} \oplus x_{15}) \times 10^{-12}, 10) + 1$$

$$b_4 = mod((x_{10} \oplus x_{12} \oplus x_{14} \oplus x_{16}) \times 10^{-12}, 10) + 1.$$

4. Each chaotic sequence X_i is generated using initial values $v_i \in (0,1)$ and control parameters a_i and b_i. The complete key set is $\{v_1, a_1, v_2, a_2, v_3, a_3, v_4, a_4, b_1, b_2, b_3, b_4\}$. Due to the high sensitivity of the chaotic map, even minimal changes in these values lead to drastically different encrypted outputs, ensuring strong key sensitivity and a large key space.

6.4.2 Pixel Permutation

In the initial stage of the encryption process, the spatial arrangement of image pixels is randomized to eliminate inherent structural correlations present in the original image. This is

accomplished by generating a chaotic sequence X_1 using the 2D CCM. The sequence is then sorted in ascending order, and the resulting index positions are used to define a permutation pattern. This index effectively reshuffles the pixel locations across the entire image, ensuring that the positional relationships between adjacent pixels are thoroughly disrupted. By applying this transformation before any intensity-based encryption, the algorithm significantly enhances resistance to statistical and structural attacks.

Steps:

1. Generate $X_1 \in [0,1]$ of size $M \times N$ using a_1 and b_1.
2. Flatten I to a 1D vector and sort it based on the index of X_1.
3. Reshape the permuted vector into a matrix A_1 as

$$A_1 = reshape\left(I\left(sortIndex\left(X_1\right)\right), M, N\right).$$

6.4.3 DNA-Based Encoding

In the proposed NIEA, DNA-inspired encoding is strategically embedded within both the permutation and diffusion stages to significantly reinforce the overall security of the encryption process. This integration creates a synergistic effect by merging biologically inspired data transformation with the high entropy of a dynamically controlled chaotic system. The use of DNA encoding introduces nonlinear and nontraditional transformations that complicate any attempt at cryptanalysis. The inherent unpredictability of DNA rules, especially when applied in conjunction with chaotic sequences, greatly enhances the system's ability to withstand differential attacks and makes unauthorized decryption challenging. Moreover, when combined with additional encoding techniques, DNA-based processing further obfuscates the internal structure of the encrypted data, rendering reverse-engineering impractical. During this phase of the algorithm, the pixel values of the intermediate image matrix A_1 are first converted into binary form. These binary sequences are then encoded using a set of predefined DNA rules, which map bit patterns to nucleotide sequences. The selection of DNA rules is not static; instead, it is governed by a separate chaotic sequence X_2 generated by the 2D CCM. This sequence dynamically determines which rule is applied to each pixel segment, thereby ensuring that the encoding process is adaptive, randomized and strongly key-dependent.

Steps:

1. Convert A_1 into 8-bit binary strings.
2. Generate X_2 using a_2 and b_2 and quantize to rule indices $Y_2 \in \{1,2,...,8\}$ as

$$Y_2 = \text{floor}\left(\left(\mathrm{X}_2(i) \times 10^{12}\right) mod\ 8\right),\ i = 1,2,\cdots, M \times N.$$

3. For each pixel, apply a DNA rule based on Y_2 and reconstruct an encoded matrix A_2.

This step increases confusion in the data without changing the overall size.

6.4.4 Pixel Diffusion

The diffusion stage is designed to modify pixel intensity values in a way that amplifies the impact of even minimal changes in the input image, thereby enforcing the avalanche effect, a key requirement for secure encryption schemes. In this stage, the objective is to ensure that a small alteration in the plaintext results in widespread and unpredictable changes throughout

the ciphertext. To achieve this, a third chaotic sequence, denoted as X_3, is generated using the 2D CCM. This sequence serves as the driving factor for the diffusion process. Specifically, an XOR-based operation is employed, where each pixel value is combined with elements from X_3. This operation introduces high nonlinearity and ensures that the resulting pixel values are not directly traceable to their original form. The dynamic nature of the chaotic sequence, coupled with the bitwise XOR operation, significantly boosts the sensitivity, unpredictability and diffusion strength of the encryption process.

Steps:

1. Generate $X_3 \in [0,1]$ of size $M \times N$ using a_3 and b_3.
2. Scale to $0-255$ and convert to integers Y_3 as follows:

$$Y_3 = \text{floor}\left(\left(\mathrm{X}_3(i) \times 10^{12}\right) mod\ 256\right),$$

3. For each value, apply a DNA rule based on Y_2 and reconstruct an encoded matrix Y_4.
4. Compute:

$$A_3 = A_2 \oplus Y_4.$$

where $\oplus$ represents the bitwise DNA XOR operation.

6.4.5 Bit-Level Reversion

The final stage of the encryption process further strengthens diffusion and complexity at the binary level. In this step, the previously generated chaotic sequence is reversed, creating an additional layer of unpredictability. This reversed sequence is then used to perform a second XOR operation on the already-diffused pixel values. By introducing this final bit-level manipulation, the algorithm ensures that even low-level binary patterns are thoroughly obfuscated, effectively eliminating any residual structure and maximizing the encryption strength. This stage contributes to both the avalanche effect and key sensitivity, ensuring that the ciphertext remains highly resistant to statistical and differential attacks.

Steps:

1. Generate X_4 using new chaotic parameters a_4 and b_4.
2. Scale to $0-255$ and convert to integers X_5 as follows:

$$X_5 = \text{floor}\left(\left(\mathrm{X}_4(i) \times 10^{12}\right) mod\ 256\right),$$

3. Convert the values of sequence X_5 into binary values.
4. Reverse the bits of the binary sequence to obtain $Z = reshape(X_5 rev, M, N)$.
5. Final encryption is performed as follows:

$$A_4 = A_3 \oplus Z.$$

6.4.6 Decryption Process

The decryption process in the NIEA is designed to precisely reverse each transformation applied during encryption, ensuring accurate recovery of the original image. Using the same initial conditions and control parameters derived through the SHA-256-based key generation mechanism, the 2D CCM regenerates the identical chaotic sequences used in the encryption phase. The

process begins by applying the inverse bit-level reversion, followed by reverse pixel diffusion using the corresponding chaotic sequence to undo the XOR-based modifications. Next, DNA decoding is performed using the same dynamically selected rules to convert the DNA-encoded data back to binary form. Finally, the inverse pixel permutation is applied by reconstructing the original spatial arrangement of the pixels based on the permutation index. Each decryption stage is tightly coupled to the chaotic sequences, ensuring that any deviation in the key or parameters results in failed reconstruction, thus preserving the system's high sensitivity and security.

ENCRYPTION ALGORITHM

Input: Color image $I \in \mathbb{R}^{M\times N\times 3}$, secret keys $\{v_1,a_1,v_2,a_2,v_3,a_3,v_4,a_4,b_1,b_2,b_3,b_4\}$.

1: *Convert image* I *to* $1D$ *sequence of pixels*
2: *Generate chaotic sequence* X_1 *using* (a_1,b_1)
3: *Sort* $X_1 \rightarrow$ *obtain permutation index* idx_1
4: *Apply pixel permutation*: $A_1 = reshape(I(idx_1),M,N)$
5: *Generate chaotic sequence* X_2 *using* (a_2,b_2)
6: *Quantize* X_2 *to form DNA rule vector* $Y_2 \in [1,8]$
7: *Convert* A_1 *to* 8*-bit binary* (A_1_bin)
8: *Convert* A_1_bin *to DNA encoded matrix* A_2 *using* Y_2
9: *Generate chaotic sequence* X_3 *using* (a_3,b_3)
10: *Quantize* X_3 *to* $0-255 \rightarrow matrix\ Y_3$
11: Convert Y_3 *to DNA encoded matrix* Y_4 *using* Y_2
12: *Diffuse*: $A_3 = A_2 \oplus Y_4$
13: *Generate chaotic sequence* X_4 *using* (a_4,b_4)

$$X_5 = \text{floor}\left(\left(X_4(i)\times 10^{12}\right) mod\ 256\right),$$

14: *Quantize and reverse*

$$matrix\ Z = reshape(X_5 rev, M, N)$$

15: *Final encryption*: $A_4 = A_3 \oplus Z$
Output: Encrypted image A_4

6.5 EXPERIMENTAL RESULTS AND SECURITY ANALYSIS

To evaluate the performance and security of the NIEA, a series of experiments was conducted and tested on the standard image dataset USC SIPI [35]. Various cryptographic metrics and visual analysis techniques were applied to assess the algorithm's resistance against statistical, differential and brute-force attacks. All experiments were conducted in MATLAB on a standard PC with an MacOS M1 and 8 GB RAM.

6.5.1 Visual Encryption Quality

To evaluate the visual effectiveness of the proposed NIEA, a side-by-side comparison between the original and encrypted images was conducted. The encrypted images appear highly randomized and noise-like, lacking any identifiable structure or visual cues from the original content. This outcome demonstrates the algorithm's strong ability to achieve confusion and diffusion, two essential cryptographic principles. As shown in Figure 6.7, there is a complete

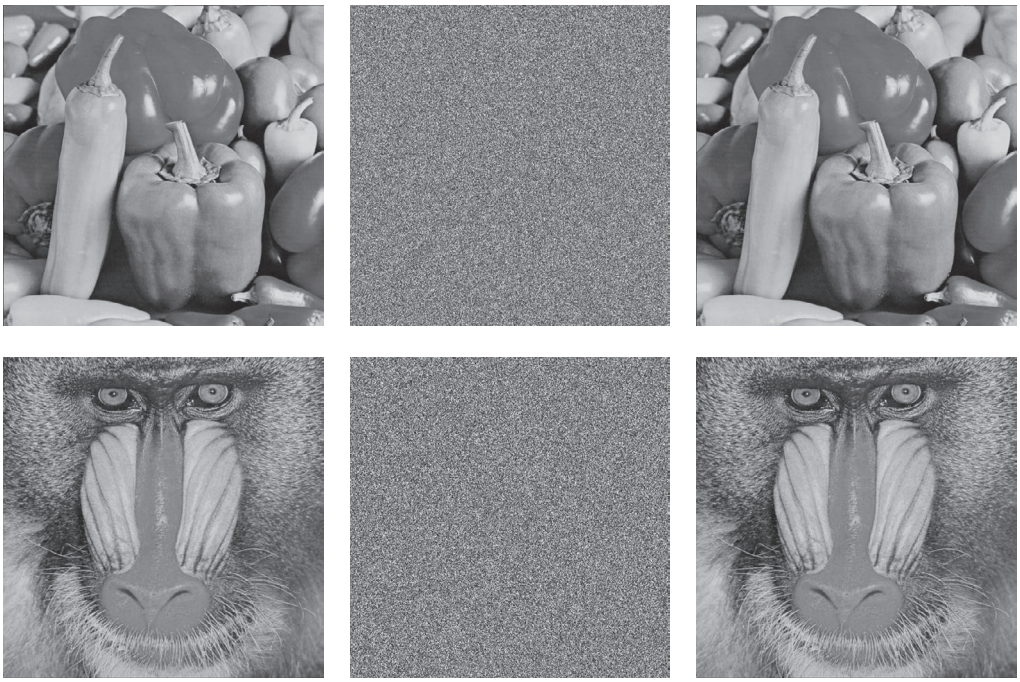


Figure 6.7 Plain, encrypted and decrypted images (512 × 512) of Pepper and Baboon, respectively.

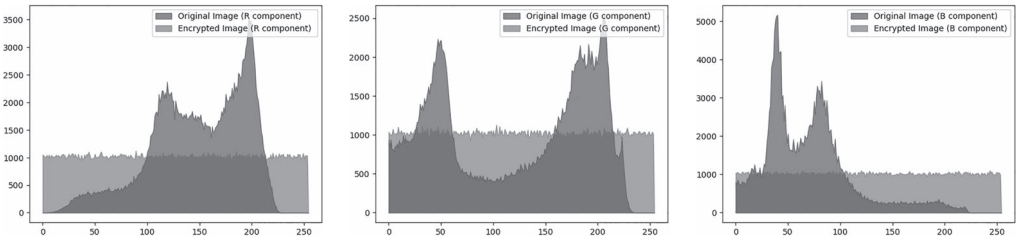


Figure 6.8 Histogram of RGB channels of the original and encrypted pepper image ($512 \times 512 \times 3$).

absence of visual correlation between the encrypted and original images, thereby rendering unauthorized interpretation or reconstruction practically impossible.

6.5.2 Histogram Analysis

To evaluate the statistical robustness of the encryption process, we first analyze the histograms of both the original and encrypted images. While the histogram of the original image displays pronounced peaks reflecting uneven pixel distributions, the encrypted image's histogram is nearly flat, indicating that pixel intensities are uniformly spread across the entire range of [0, 255] (see Figure 6.8). This uniformity effectively conceals inherent statistical features, mitigating the risk of leakage that adversaries could exploit. Natural digital images typically exhibit heterogeneous pixel distributions with recognizable patterns and intensity concentrations. A resilient image encryption scheme must be capable of eliminating these unique statistical features to resist attacks based on pixel distribution analysis. To test the

effectiveness of our NIEA in this regard, we applied it to encrypt two standard color test images. The results reveal that NIEA transforms initially irregular pixel distributions across all three-color channels into uniformly random distributions. This transformation highlights NIEA's ability to disrupt spatial redundancy and diminish recognizable patterns, thereby significantly enhancing resistance to statistical and distribution-based attacks.

To further quantify this uniformity, we performed a chi-square test on the encrypted images. The test statistic is computed using the following equation:

$$\chi^2 = \sum_{\beta=1}^{Q} \frac{\gamma_\beta - M \times N \times \delta}{M \times N \times \delta}.$$

In this equation, γ_β signifies the quantity of ciphertext pixels possessing a value of $\beta - 1$. The variable Q denotes the maximal quantity of pixel values. For an 8-bit image, $Q = 2^8$ and $\delta = 1/Q$. Furthermore, M and N represent the vertical and horizontal dimensions of the encrypted image, respectively. Subsequently, one can ascertain the threshold value $\chi^2 0.05(255)$ of the chi-square test at a significance level of 0.05, which computes to 293.2478 [6]. If the calculated chi-square value for an encrypted image falls below this threshold, the image can be considered statistically uniform with respect to pixel intensity distribution. As presented in Table 6.3, all test images encrypted using NIEA pass the chi-square test, affirming the algorithm's capability to produce ciphertext images with high statistical randomness. This result further confirms the effectiveness of NIEA in neutralizing the distinct pixel distribution characteristics inherent to unencrypted images.

6.5.3 Correlation Coefficient Analysis

The degree of correlation between an original image and its encrypted counterpart can be quantitatively assessed using the correlation coefficient, which serves as a measure of linear dependency between corresponding pixel values. This coefficient is calculated using the following expression:

$$\text{Correlation}[v_1, v_2] = \frac{Co[v_1, v_2]}{\rho[v_1] \times \rho[v_2]}$$

where v_1 and v_2 denote the pixel intensity values of the original and encrypted images, respectively, $Co[v_1, v_2]$ is the covariance between the two; $\rho[v_1]$ and $\rho[v_2]$ are the corresponding standard deviations. Ideally, in a secure encryption system, the correlation between the original and encrypted images should be minimal, approaching zero, which implies that the ciphertext bears no statistical resemblance to the plaintext. The correlation coefficients were evaluated in three spatial orientations—horizontal, vertical and diagonal to provide a comprehensive analysis of statistical independence between neighboring pixels. The results, as summarized in Table 6.4,

Table 6.3 Results of the Chi-Square Test Obtained for NIEA

Image size	*Test image*	χ^2 *value threshold (293.2478)*	*Passed or not*
512×512	Barbara	222.26	Passed
512×512	Pepper	254.01	Passed
512×512	Lena	234.12	Passed

Table 6.4 Comparative Outcomes of the Correlation Coefficient Values of Plain and Encrypted Pepper Image in the Horizontal, Diagonal and Vertical Directions

Schemes	*H*	*D*	*V*
Plain	0.9894	0.9804	0.9680
Proposed	−0.0012	−0.0019	0.00010
[5]	−0.0015	0.0067	0.00015
[11]	0.0043	−0.0054	−0.0018
[8]	−0.0011	0.0046	0.0018
[12]	−0.0009	0.00175	−0.0004

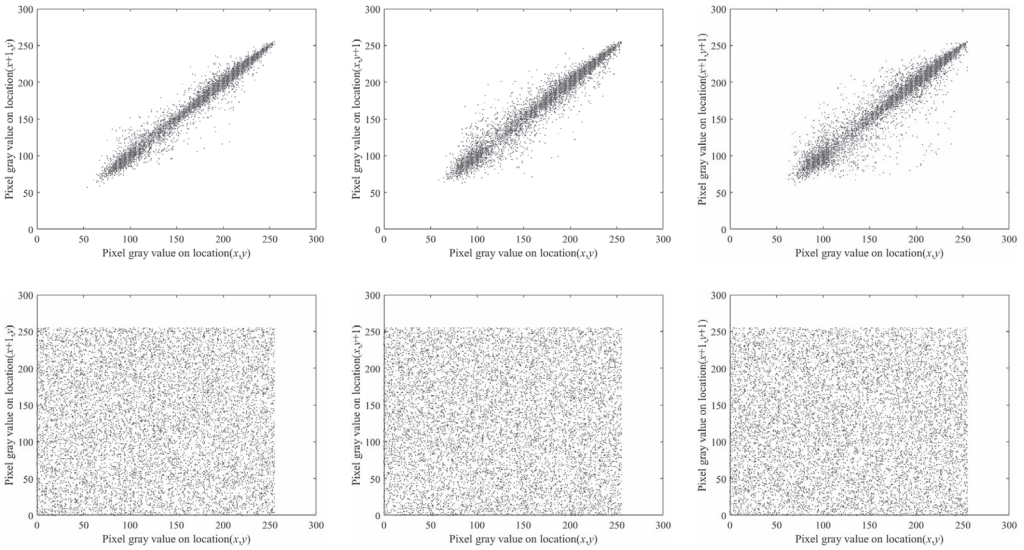


Figure 6.9 Pixel correlation assessment of the plain and encrypted pepper image.

indicate that all correlation values are very close to zero, suggesting a high level of decorrelation. This observation is further visualized in Figure 6.9, reinforcing the effectiveness of the proposed NIEA in eliminating linear dependencies and resisting statistical attacks.

6.5.4 Information Entropy

In the context of information theory, information entropy serves as a metric to quantify the degree of uncertainty or randomness present within a dataset. When applied to digital images, entropy reflects the unpredictability of pixel intensity values. Let I_i represent a discrete pixel value from a sequence, the entropy $E(I)$ can be defined mathematically as:

$$E(I) = \sum_{i=1}^{n} P(I_i) log_b P(I_i)$$

Here, $P(I_i)$ denotes the probability of occurrence of the pixel value I_i, and n refers to the total number of possible intensity levels, typically determined by the bit depth of the image. For an 8-bit grayscale image, where pixel values range from 0 to 255 and are uniformly

Table 6.5 Comparative Outcomes of the Entropy Values for Proposed Scheme and Various Schemes

Scheme	*Entropy value*
Proposed	7.9998
[5]	7.9989
[11]	7.9990
[8]	7.9989
[12]	7.9988

distributed, the ideal entropy value is 8. This theoretical maximum indicates that all pixel values occur with equal likelihood, implying maximum randomness, an essential property for secure encrypted images. The entropy values obtained from the NIEA, along with those from other comparative schemes, are reported in Table 6.5. These results demonstrate the efficacy of the NIEA in achieving near-optimal entropy, reinforcing its strength in producing highly randomized ciphertexts.

6.5.5 NPCR and UACI Tests

To assess the robustness of an encryption algorithm against differential attacks, two key statistical metrics are widely employed: the Number of Pixel Change Ratio (NPCR) and the unified average changing intensity (UACI). These metrics measure the sensitivity of the encryption process to slight variations in the input (plaintext) image. Ideally, a secure encryption scheme should yield NPCR values close to 100% and UACI values around 33.3%, indicating a high level of unpredictability in the encrypted output. Let F_1 and F_2 be two cipher images generated from two plaintext images that differ only by a single pixel. A secure system should produce encrypted outputs where even such a minor change in input leads to significantly different ciphertexts. The NPCR value quantifies the percentage of differing pixels between F_1 and F_2 and is computed as follows:

$$\text{NPCR} = \frac{1}{M \times N} \times \sum\nolimits_{i,j} D(i,j) \times 100.$$

where $M \times N$ is the total number of pixels in the image, $D(i,j)$ is defined as:

$$D(i,j) = \begin{cases} 0, if\ F_1(i,j) = F_2(i,j) \\ 1,\ if\ F_1(i,j) \neq F_2(i,j). \end{cases}$$

In contrast, the UACI metric evaluates the average intensity variation between the two cipher images and is calculated using the following expression:

$$\text{UACI} = = \frac{1}{M \times N} \times \sum\nolimits_{i,j} \frac{\left| C_1(i,j) - C_2(i,j \right|}{M},$$

Here, M denotes the maximum possible pixel intensity value (typically 255 for 8-bit images),

Table 6.6 Comparative Results of NPCR (%) Values for Proposed Scheme with Existing Algorithms

RGB	*Proposed*	*[5]*	*[11]*	*[8]*	*[12]*
R	99.61	99.60	99.62	99.57	99.60
G	99.60	99.61	99.60	99.60	99.61
B	99.61	99.62	99.61	99.59	99.62

Table 6.7 Comparative Results of UACI (%) Values for Proposed Scheme with Existing Algorithms

RGB	*Proposed*	*[5]*	*[11]*	*[8]*	*[12]*
R	33.38	32.18	30.33	32.12	32.08
G	33.48	30.39	30.36	30.723	30.43
B	33.20	30.24	31.50	30.701	32.46

$C_1(i,j)$ and $C_2(i,j)$ represent the pixel intensities at position (i,j) in the respective encrypted images. These metrics collectively help in evaluating the sensitivity and diffusion characteristics of an image encryption algorithm. Tables 6.6 and 6.7 provide a comparative analysis of NPCR and UACI values for the NEIA in comparison with recent state-of-the-art methods using the standard Pepper image. The results confirm the proposed algorithm's strong resistance to differential attacks by consistently achieving values close to the theoretical optima.

6.5.6 Key Space

To ensure that a cryptographic system is resilient against brute-force attacks, it is essential to maintain an adequately large key space. In chaotic-based encryption, this security requirement is further strengthened by the system's extreme sensitivity to initial conditions and control parameters, which makes it computationally infeasible to guess the correct key through exhaustive search. In the proposed image encryption framework, a 256-bit external secret key is utilized. This key is then processed, often through a cryptographic hash like SHA-256, to derive twelve critical parameters: $v_1, v_2, v_3, v_4, a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$. These values function as the initial conditions and control parameters for the 2D CCM, which forms the cryptographic core of the system.

Each of these parameters is assumed to be a real number represented with a precision of 15 decimal places (i.e., double precision). Since each parameter spans a continuous range and offers roughly 10^{15} distinct values, the effective key space can be estimated as

$\approx 2^{256}.10^{15\times12} \approx 2^{256}.10^{180} \approx 2^{256}.2^{180log_2(10)} \approx 2^{256}.2^{180\times3.3219} \approx 2^{256}.2^{597.60}$. So, the total key space becomes $2^{256+598} = 2^{854}$.

Table 6.8 provides a comparative analysis of key space values for the NEIA in comparison with recent state-of-the-art methods.

6.5.7 Key Sensitivity

The security strength of an encryption algorithm is heavily influenced by its sensitivity to the secret key. An effective cryptosystem should exhibit high key sensitivity, meaning that even an infinitesimal change in the secret key should result in a completely different decryption

Table 6.8 Comparison of Key Space

Scheme	*Key Space*
Proposed	2^{854}
[36]	2^{128}
[37]	2^{299}
[34]	2^{200}
[38]	2^{744}



Figure 6.10 Key sensitivity: Encrypted and decrypted pepper image with original key and decrypted pepper image with key 2 (with slight change in x_0 as $x_0 + 10^{-15}$) from left to right.

output. Consequently, any slight alteration in the key, whether in the initial conditions or control parameters, should render the decryption process ineffective, producing an output that bears no resemblance to the original image. To validate this property, we conducted an experiment using the standard Pepper image. The image was encrypted using a correctly configured secret key, and the resulting ciphertext is illustrated in Figure 6.10. Next, the ciphertext was decrypted using a slightly modified key, where the initial condition x_0 was altered by a negligible amount, specifically $x_0 + 10^{-15}$. The decryption output, however, failed to reconstruct the original image and instead produced a visually distorted, noise-like result, lacking any discernible content from the plaintext. This experiment confirms that the NIEA ensures strict key dependency, whereby only the exact original secret key can correctly decrypt the ciphertext. Any deviation, even at the scale of 10^{-15}, leads to decryption failure, further demonstrating the algorithm's robustness against key sensitivity-based attacks.

6.5.8 Plaintext Attack Analysis

The conventional cryptanalysis techniques encompass several well-known methods, including known-plaintext attacks, ciphertext-only attacks, chosen-ciphertext attacks and Chosen Plaintext Attack (CPA). Among these, the CPA is widely regarded as the most challenging and formidable threat model. Consequently, if an encryption scheme can successfully defend against CPA, it is generally assumed to offer resilience against the other three categories of attack as well. A common approach employed by attackers in such scenarios is to utilize plaintexts consisting entirely of uniform values, such as all-white or all-black images, to exploit structural weaknesses in the cipher. Therefore, these uniform images serve as reliable benchmarks for evaluating the system's resistance to CPAs.

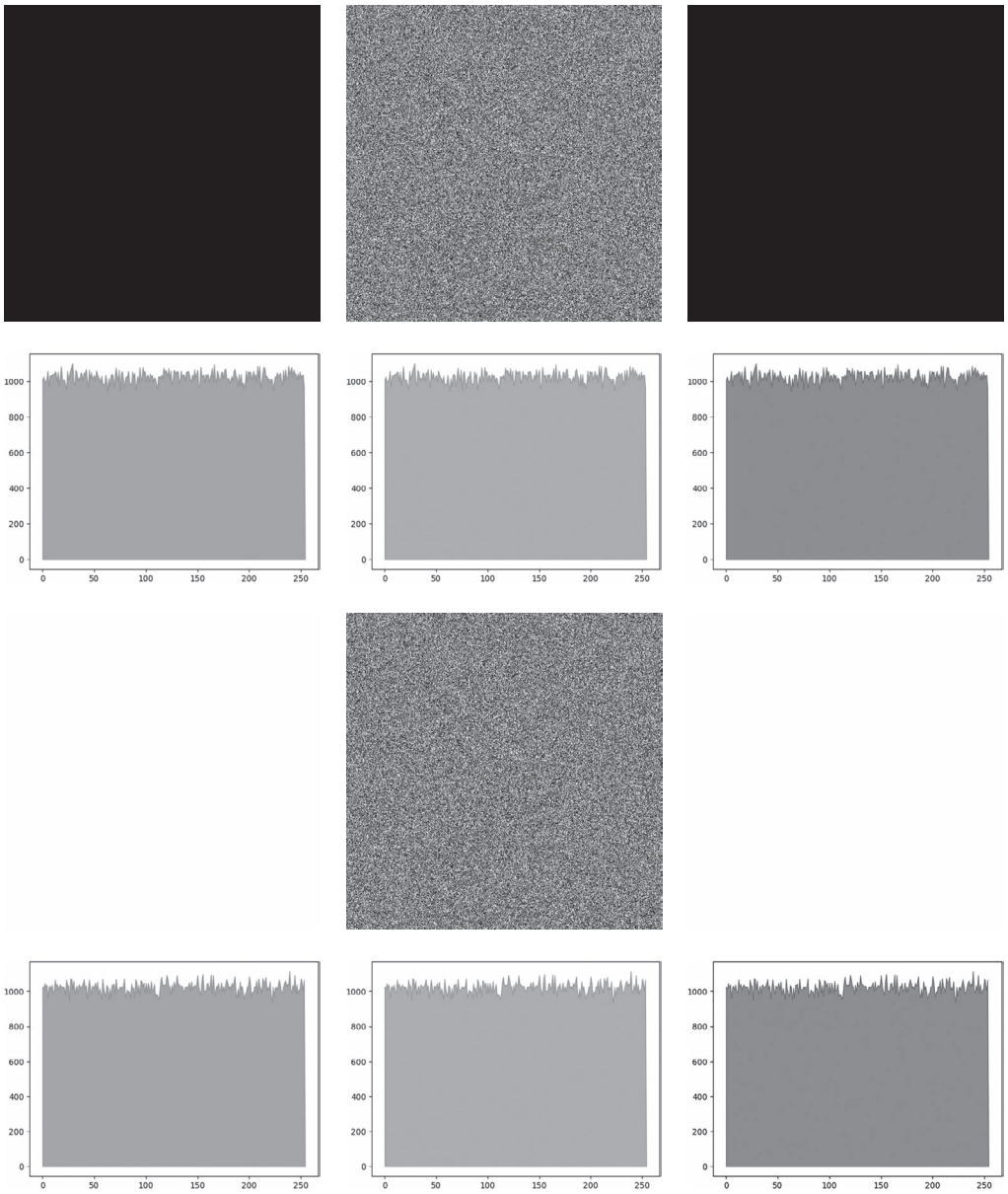


Figure 6.11 Experimental results of NIEA against chosen plaintext attacks: Plain, encrypted and decrypted black image (first row) and histogram of RGB channels of black (second row). Plain, encrypted and decrypted white image (third row) and histogram of RGB channels of white image (fourth row).

In the context of this evaluation, both a white image and a black image were encrypted using the NIEA. The resulting cipher images, presented in Figure 6.11, appear entirely noise-like and unrecognizable, with uniform histogram distributions, indicating strong obfuscation of any input patterns. This visual evidence confirms that the encryption process effectively masks any plaintext regularities. Further analysis, as summarized in Table 6.4, reveals that the pixel correlation coefficients in all spatial directions (horizontal, vertical and diagonal) are significantly reduced, approaching values near zero. Additionally, the entropy values of the encrypted

outputs are consistently close to the theoretical maximum for 8-bit images (see Table 6.5). These findings collectively affirm the ability of the NIEA to robustly withstand CPAs, achieving performance metrics that align closely with ideal security benchmarks.

6.5.9 Noise Attack Analysis

To evaluate the resilience of the NIEA against noise-based attacks, a structured test procedure was implemented. Initially, the original image I was encrypted using the NIEA. Subsequently, various types of noise, including Gaussian noise and salt-and-pepper noise, were introduced into the encrypted image at controlled noise levels. Specifically, Gaussian noise with a 2% variance and salt-and-pepper noise at a 5% density were applied. Following this, the noise-corrupted ciphertexts were decrypted in an attempt to reconstruct the original image, yielding output image O. The visual results of this process are illustrated in Figure 6.12, while the quantitative evaluation is provided in Table 6.9.

To assess the degradation introduced by the noise and the system's capacity to recover the original content, the mean-squared error (MSE) metric is used. MSE quantifies the average squared difference between the pixel values of the original image I and the decrypted image O and is mathematically expressed as:

$$\mathrm{MSE} = \frac{1}{M \times N} \sum_{i=1}^{M} \sum_{j=1}^{N} \left(I(i,j) - O(i,j) \right)^2,$$

where M and N represent the total number of rows and columns in the image, respectively.

The experiment confirms that, even in the presence of moderate noise, the NIEA retains a high degree of robustness, as evidenced by the manageable MSE values and the visual integrity observed in the reconstructed images.

6.5.10 Mean Absolute Error

An additional quantitative metric frequently employed to assess the resilience of image encryption schemes against differential attacks is the mean absolute error (MAE). This metric evaluates the average absolute difference between the pixel values of a plaintext image and its corresponding ciphertext image. Specifically, it captures how significantly the encryption process alters the original pixel intensities.

The MAE for an image of dimensions $M \times N$, where $P(i,j)$ and $E(i,j)$ represent the pixel values at position (i,j) in the plaintext and encrypted images, respectively, is defined as:

$$\mathrm{MAE} = \frac{1}{M \times N} \sum_{i=1}^{M} \sum_{j=1}^{N} \left(P(i,j) - E(i,j) \right).$$

Table 6.9 Comparative Results of MSE Values for NIEA and Existing Algorithms

Images	*Proposed*	*[36]*	*[8]*	*[38]*
Lena	7619.18	8888.8	10869.73	8966.97
Baboon	7816.12	—	—	—
Pepper	9012.22	10092.3	—	10049.4
Black	17018.98	—	—	—
White	18027.22	—	—	—

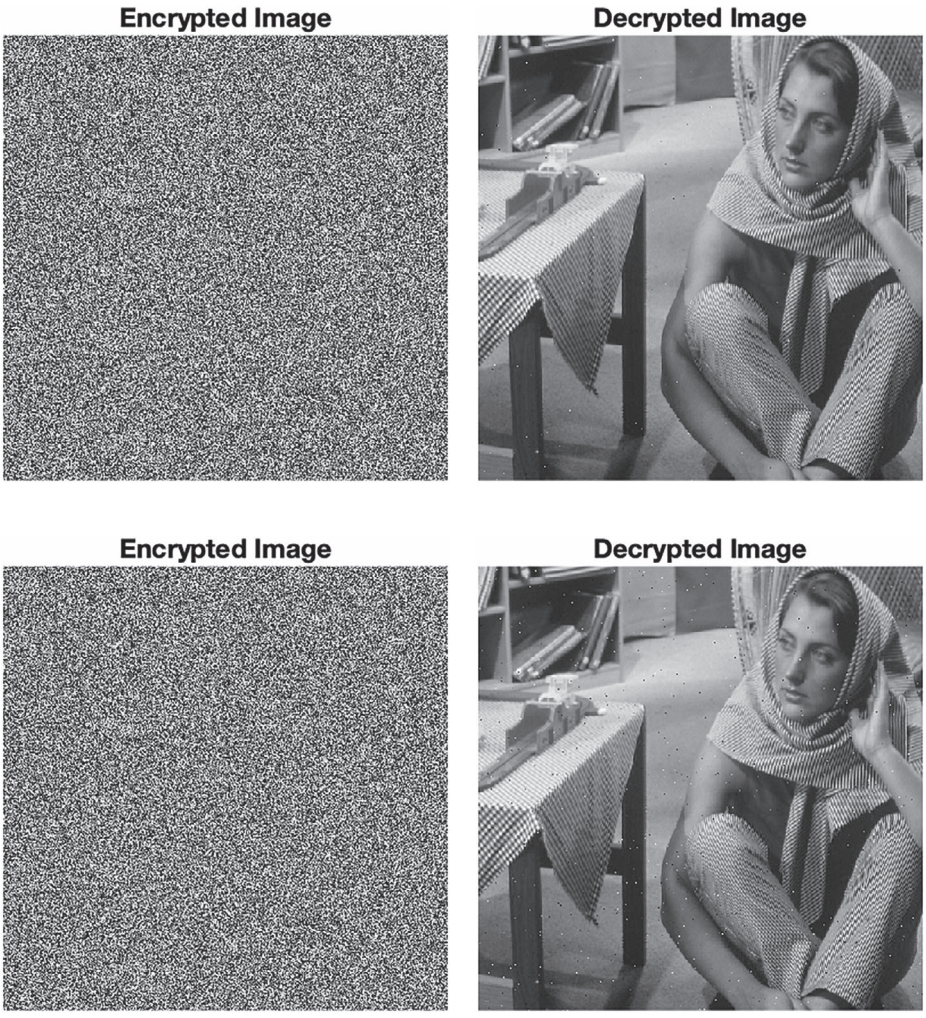


Figure 6.12 NIEA under 2% Gaussian noise and 5% Salt-and-Pepper noise; decrypted images remain close to the original, showing strong robustness.

A higher MAE value implies a greater deviation between the plaintext and the ciphertext, which is desirable as it reflects strong diffusion and high sensitivity to input changes, key indicators of a secure image cryptosystem. The MAE values computed for the proposed NIEA are summarized in Table 6.10, alongside comparative results reported for other recent algorithms. Across all test images, the NIEA consistently achieves MAE values that either are comparable to or exceed those of competing methods, thereby reinforcing its effectiveness in resisting differential attacks.

6.5.11 The Peak Signal-to-Noise Ratio

To evaluate the visual dissimilarity between plaintext and encrypted images, two widely used metrics are employed: the peak signal-to-noise ratio (PSNR) and the structural similarity index measure (SSIM). These metrics quantify the extent to which the encryption process has altered

Table 6.10 Comparative Experimental Results of MAE Values

Images	*Proposed*	*[36]*	*[8]*	*[38]*
Lena	82.270	77.3752	87	77.7932
Baboon	78.891	—	—	—
Pepper	84.781	81.7740	—	81.8995

the image content. The PSNR provides an inverse relationship with the MSE; as the MSE between the original and encrypted image increases, the PSNR decreases. Mathematically, the PSNR is expressed as:

$$\text{PSNR} = 20 \times log_{10} \frac{P_{max}}{\sqrt{\text{MSE}}},$$

where P_{max} denotes the maximum possible pixel intensity value (typically 255 for 8-bit images). On the other hand, the SSIM metric assesses the perceived structural similarity between the original and encrypted images by analyzing aspects such as luminance, contrast and texture. In the context of encryption, lower values of both PSNR and SSIM indicate a higher degree of distortion, which is desirable, as it implies the ciphertext carries minimal perceptible resemblance to the original image. For the proposed NIEA, the computed PSNR and SSIM values are summarized in Table 6.11. The results show that the SSIM values approach zero, while the PSNR values remain low across all tested images. These outcomes confirm the algorithm's capability to significantly alter structural and intensity information, thereby enhancing the security of the encryption process by reducing the visual similarity between the plaintext and the ciphertext.

6.5.12 Data Loss Attack Analysis

To test the robustness of the encryption algorithm against data loss attacks, a portion of the encrypted image data was intentionally removed (cropped), with the missing areas replaced by zeros. The plaintext image was then reconstructed from these cropped versions of the encrypted image. The results for images with 1/16 data crop and 1/4 data crop are shown in Figure 6.13.

To evaluate the quality of the recovered images, the MSE and PSNR were calculated for each case. A higher PSNR value indicates greater resilience of the cryptosystem to data loss and improved recovery accuracy.

The experimental results for MSE and PSNR are summarized in Tables 6.9 and 6.11. The results demonstrate that the NIEA effectively withstands various types of data loss attacks, maintaining good recovery performance in the spatial domain, as evidenced by the figures and tables presented.

6.5.13 The National Institute of Standards and Technology Analysis

The National Institute of Standards and Technology Analysis (NIST) SP 800 statistical test suite, developed by the NIST, is a widely recognized method for evaluating the randomness quality of cryptographic outputs, particularly those derived from PRNGs. This suite comprises a series of tests applied to binary sequences to determine whether they exhibit characteristics expected of truly random data.

To successfully pass any of these tests, the associated p-value must exceed a significance threshold of 0.01. In this study, the proposed NIEA was subjected to the NIST evaluation

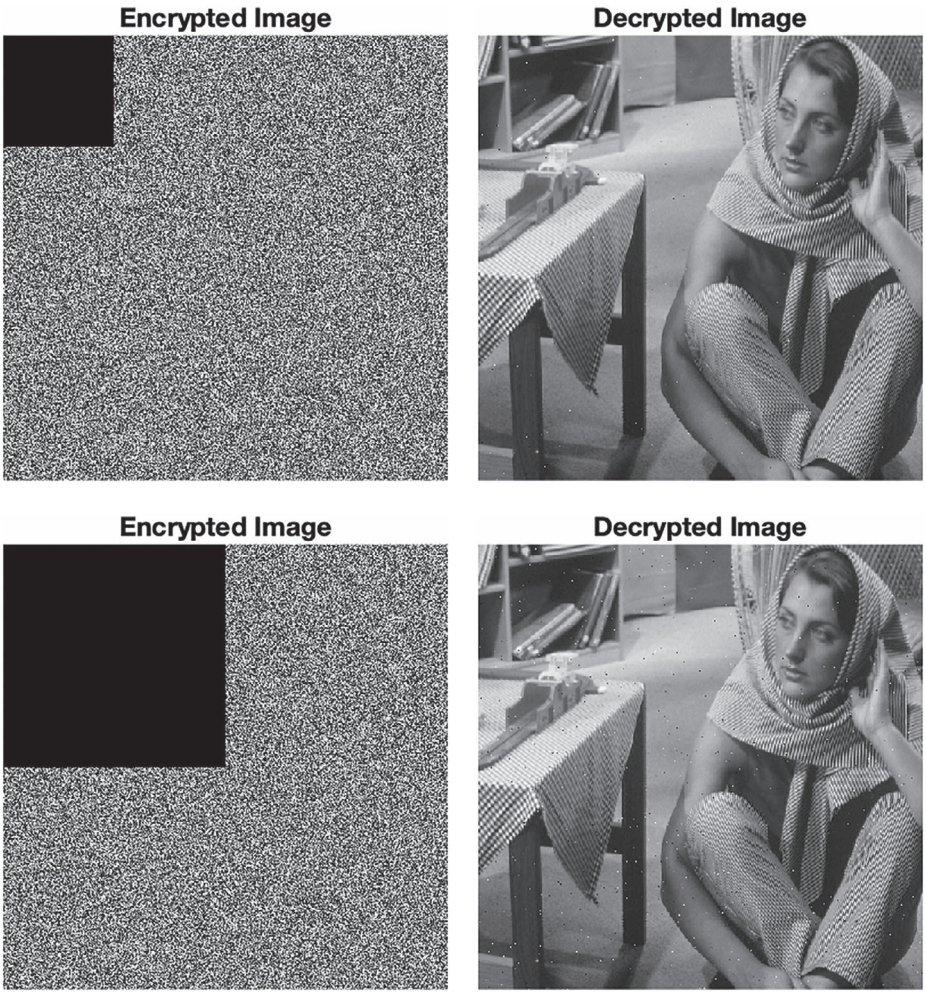


Figure 6.13 Experimental results for NIEA against data loss attack.

Table 6.11 PSNR and SSIM Experimental Results of Four Encryption Algorithms

	PSNR				*SSIM*
Images	*Proposed*	*[36]*	*[8]*	*[38]*	*Proposed*
Lena	8.7891	8.64233	7.7677	8.60435	0.0084
Baboon	8.8981	—	—	—	0.0082
Pepper	8.9811	8.09089		8.10939	0.0068
Black	4.9822	—	—	—	0.0038
White	4.2899	—	—	—	0.0048

by converting a large encrypted image into a bitstream through row-wise concatenation of pixel values. The analysis was conducted separately on the red, green and blue channels of a standard 512 × 512 Pepper image, and the results are summarized in Table 6.12. All p-values obtained from the tests comfortably exceed the 0.01 threshold, indicating that the encrypted bitstreams exhibit strong randomness characteristics. This outcome confirms that the NIEA

Table 6.12 Results of NIST Test on the RGB Channels of Encrypted Pepper Image

Test name	*Red*	*Green*	*Blue*	*Remarks*
Frequency	0.201271	0.786750	0.278282	Success
Block frequency	0.782940	0.987610	0.162934	Success
Run	0.282760	0.279890	0.290166	Success
Long runs of ones	0.236766	0.786634	0.273699	Success
Rank	0.109873	0.214566	0.090570	Success
Spectral FFT	0.675439	0.765509	0.875391	Success
Nonoverlapping	0.267844	0.990971	0.182901	Success
Overlapping	0.354567	0.087923	0.482610	Success
Serial	0.098711	0.089271	0.912220	Success
Approximate entropy	0.278651	0.187201	0.789901	Success
Cum. sums forward	0.174566	0.198276	0.827901	Success
Linear complexity	0.109865	0.981630	0.119029	Success

Table 6.13 Time (s) Required by Various Algorithms to Encrypt Images

Algorithms	*Size*	*Resolution*	*Time*
Proposed scheme	$512\times512\times3$	512×512	0.181
Ref. [5]	$256\times256\times3$	256×256	0.62
Ref. [11]	$256\times256\times3$	256×256	0.75
Ref. [8]	$256\times256\times3$	256×256	0.02
Ref. [12]	$256\times256\times3$	256×256	0.09

meets the statistical criteria set by NIST, further validating the cryptographic robustness and reliability of the proposed encryption method.

6.5.14 Efficiency Analysis

The execution time of an image encryption algorithm is a crucial factor when considering its applicability to real-time systems and computationally constrained environments. To ensure the proposed NIEA is both efficient and secure, its time complexity has been thoroughly evaluated. The encryption process is structured into four main components: pixel permutation to disrupt spatial relationships, DNA-inspired encoding to introduce nonlinearity, pixel diffusion to spread local changes globally and bit-level reversion for added complexity. Each of these steps processes every pixel of the input image individually. Letting M and N denote the number of rows and columns, respectively, the operations scale proportionally with the number of pixels $(M\times N)$. The chaotic sequence generation, central to all stages, is based on iterative updates using the proposed 2D CCM. This operation also exhibits linear growth with respect to image size. As each stage requires no more than a single pass over the image, the cumulative time complexity of the entire encryption procedure is: $O(M\times N)$. This confirms that the algorithm scales linearly and can accommodate larger images without introducing high computational cost, making it particularly suitable for practical deployments. A comparative summary of execution times between NIEA and other recent image encryption methods is provided in Table 6.13, which highlights the algorithm's competitive performance and low computational overhead.

6.6 CONCLUSION AND FUTURE SCOPE

This chapter presents an NIEA grounded in a custom-designed 2D CCM. The encryption framework is structured into four sequential stages: pixel permutation, DNA-based encoding, pixel diffusion and bit-level reversion. Each phase is independently driven by chaotic sequences derived from the 2D CCM with unique control parameters allocated to every stage. This strategy significantly increases both the complexity and cryptographic strength of the system. Comprehensive dynamic analyses validate the robust chaotic behavior of the 2D CCM, characterized by high approximate entropy, uniform value distribution and notable sensitivity to initial conditions and key parameters. The effectiveness of the NIEA is further supported by experimental evaluations, which highlight uniform histogram characteristics, high entropy values, minimal pixel correlation and strong plaintext sensitivity. In addition to its strong security features, the NIEA maintains low computational complexity, making it well-suited for real-time processing and embedded or resource-limited environments. Unlike many existing chaos-based encryption techniques that often struggle to balance performance with security, the NIEA offers a compelling trade-off by combining a lightweight design with high encryption reliability. Moreover, the modular structure of the algorithm enables seamless adaptation to video encryption as well as potential integration with multimedia processing frameworks such as compression and watermarking. Future research directions may include: real-time extension of the NIEA for video encryption, incorporation of multimedia watermarking or compression-aware encryption, application of deep learning or optimization techniques to dynamically tune the chaotic parameters and improve system adaptability.

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Chapter 7

Quantum Key Distribution Protocols

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7.1 INTRODUCTION TO QUANTUM-KEY DISTRIBUTION

The technology known as quantum-key distribution (QKD) enables two parties to create a shared, secret random key, safeguarded by quantum mechanical rules. In contrast to classical cryptographic protocols, QKD provides provably secure key establishment based on the principles of physics instead of computational complexity. The basic premise is that the eavesdropping effort on the quantum channel always disturbs the system and indicates the intruder's presence. QKD finds specific applications in quantum computing, which endangers conventional cryptography. During the past decades, it has come a long way from being merely theoretical proposals to experimental demonstrations. QKD, despite facing technical impediments, promises a new era in secure communication. New innovative capabilities, such as remote monitoring, more flexibility, and lower operating and maintenance costs, may be made possible by recent developments in reactor designs. Adopting new digital technologies would enable operational concepts such as near-autonomous or semiautonomous control and two-way communications for real-time integration with the future smart electric grid. Nevertheless, such constant data transfer from and to a reactor site may provide additional difficulties and dangers, making cybersecurity a top priority. It has been demonstrated that traditional encryption techniques based on information technology, which primarily rely on computational complexity, are susceptible to cyberattacks. Almost every asymmetric encryption might be hacked with the development of quantum computing. An RSA-2048-bit key, for instance, can be breakable in eight hours. QKD's feasibility for secure communications to overcome this difficulty was investigated. Using the principles of quantum physics rather than intricate mathematics, QKD is a physical-layer security system. Since QKD makes the communication parties aware of eavesdropping, it guarantees not only complete security but also the identification of possible intrusion. A special simulation tool (NuQKD) was created to enable BB84 QKD protocol simulation in real time between two remote terminals to evaluate the hypothesis. A generic reference scenario that encompasses a variety of internal and external communication linkages to a nuclear site is put forth. Receiver operating characteristic curves were computed for several QKD setups, and the internal and external data lines were modeled using NuQKD (Gkouliaras et al., 2024). An investigation of QKD's performance showed that it can offer sufficient secret key rates with few false alarms and might potentially meet the high secrecy requirements of the nuclear sector for fiber-optic distances up to 75 km. QKD can deliver up to 21.5 kbps of secret key rate at a distance of 1 kilometer and 14.4 kbps at 10 km, according to a conservative estimate. Real data from a PUR-1 completely digital reactor was analyzed to estimate the target secret key rates for the relevant lines, which were 16 and 80 bps, respectively. Therefore, it is demonstrated that QKD is compatible with point-to-point reactor communication topologies, both current and future. Studying quantum communications for nuclear reactors is

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encouraged by these findings. In QKD, a shared private key that is only known by the sender and the recipient can be created after the error rate is sufficiently low and enough photons have been detected. This outlines the potential benefits of this innovative technology and its many applications for the critical infrastructure of hydropower plants and dams. The talks that were given might be seen as a forerunner to a blueprint for quantum cybersecurity that would identify and mitigate pertinent risks. In the simplest version of QKD, Alice and Bob, the two communicators, disseminate correlated bit-strings by quantum optical processes. Post-processing is necessary once quantum processes are finished. To produce the final keys, they need to apply several mathematical and numerical processing techniques. Data mistakes are increased if an eavesdropper (Eve) tries to intercept the quantum stage of the operation, which is shown in Figure 7.1 (Green et al., 2023).

QKD is a cutting-edge communication technique that offers the highest level of security guarantee using the basic ideas of quantum physics. The creation and implementation of a completely automated inter-building QKD framework for the safe and smooth production and distribution of cryptographic keys using the BB84 protocol is based on weak coherent pulses. Two buildings separated by 300 meters are connected using a quantum communication link through a free-space atmospheric channel. This framework is experimentally verified. An innovative synchronization method that utilizes the native NavIC constellation is created and put into practice. With a mean photon number per pulse of 0.15 and a QBER <3%, this QKD system can generate safe keys at up to 300 Kbps. The technology has been designed to simulate and assess the intercept–resend eavesdropping assault. To unconditionally secure two-way communication over Ethernet, a unique end-to-end encrypted video using a quantum security chatting application (QuViC) is also built and integrated with the QKD framework. This app works in tandem with quantum communication (Jain et al., 2022). Secure communications for a variety of sectors, including banking and defense, might be revolutionized by quantum communication networks. The majority of the research on quantum networks focuses on issues related to resource allocation and networking using software-defined networks (SDNs). However, SDNs bring with them a number of problems, including scalability problems and app modification assaults, a new method for establishing

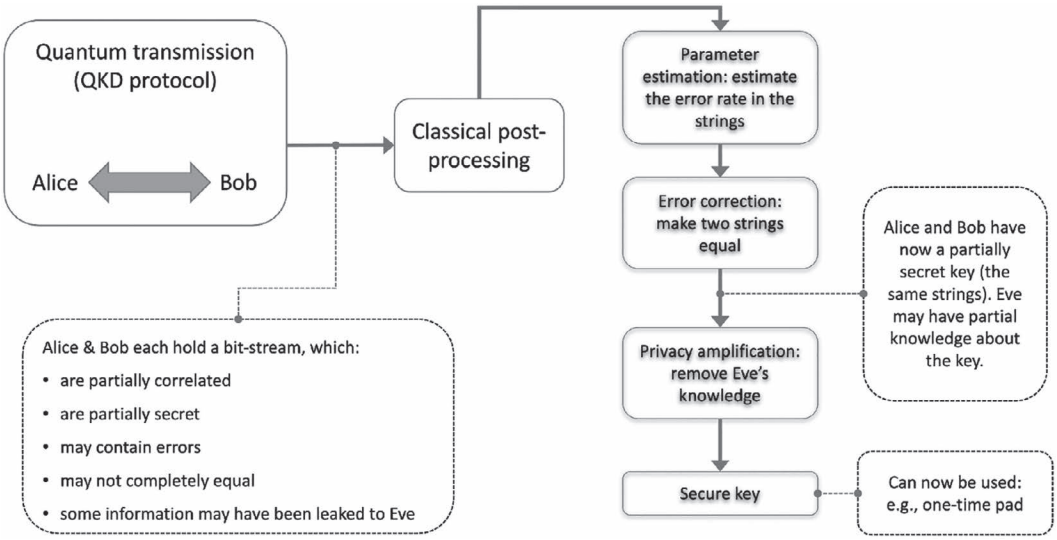


Figure 7.1 Basic depiction of quantum transmission (QKD protocol) (Green et al., 2023).

hardware-routed quantum communication networks instead of software-defined ones that use laser communications to identify qubit photons. To construct the hardware-routed quantum network, a thorough execution of the new plan is offered, and two innovative algorithms are suggested: bandwidth sharing and equitable bandwidth sharing. When compared to the First Come First Serve algorithm, the algorithms produce a crucial rate rise of 118% and enhanced network resource usage of 147% (Singal et al., 2022).

Objectives of the Chapter

The objectives of the chapter are as follows:

1. To introduce the fundamental principles and motivations behind QKD and its significance in modern cryptographic systems;
2. To explain foundational QKD protocols such as BB84 and their practical variants, including decoy-state, measurement-device-independent (MDI), and twin-field (TF) QKD;
3. To explore advanced developments in continuous-variable QKD (CV-QKD) and satellite-based QKD and their practical implementation challenges;
4. To examine QKD authentication, error correction, and privacy amplification mechanisms for secure key generation;
5. And, to discuss network architectures, routing protocols, and the integration of QKD into classical and hybrid systems for scalable quantum communications.

Organization of the Chapter

The rest of the chapter is organized as follows. Section 7.2 focuses on foundational protocols such as BB84 and its variants. Section 7.3 discusses advanced protocols, including MDI-QKD and TF-QKD, along with associated security models. Section 7.4 explores CV-QKD and satellite-based QKD, highlighting recent advancements. Section 7.5 presents key processes like authentication, error correction, and privacy amplification. Section 7.6 examines QKD network architectures, routing protocols, and integration with classical infrastructure. Section 7.7 outlines emerging trends and future directions in QKD technologies. And, finally, section 7.8 concludes the chapter.

7.2 FOUNDATIONAL PROTOCOLS: BB84 AND ITS VARIANTS

Bennett and Brassard devised the BB84 protocol in 1984, making it the oldest and most impactful QKD scheme. It encodes bits securely using quantum states of polarized photons in various bases. When an eavesdropper intercepts and measures the photons, the introduced anomalies are detectable in the key. Variants like B92 and the six-state protocol generalize BB84's ideas with alternative trade-offs between security and efficiency. Most practical realizations modify BB84 to account for real-world deficiencies, such as photon loss and detector inefficiency. These root protocols pave the way for continuing innovation in the field. They continue to be a standard to measure the security of newer QKD systems.

Among the most crucial concerns facing the world now is security awareness. Information security is the topic that introduces the concept of cryptography. Traditional security methods based on cryptography assume that keys are exchanged prior to secure connections. Safe key management is essential when incorporating cryptographic operations into any

system approach that is necessary for communicating and transferring a secret key between two entities is the most important consideration. If the basic management techniques are inadequate, the systems will be susceptible to errors and potentially lethal external attacks. Quantum cryptography is a technique for safely encrypting data transferred between parties. and identifying those attempting to listen in on the talk. Nowadays, among the world's most crucial concerns is security awareness. When discussing the topic of information security, the concept of cryptography is introduced. Security methods based on traditional cryptography assume that keys are exchanged prior to secure connections. The safe key management technique needed to convey and transfer a secret key between two entities is the most important piece of information to take into account when integrating cryptographic activities with any system. Inadequate basic management techniques will leave the systems open to errors and perhaps lethal external attacks. Quantum cryptography is a technique that encrypts data exchanged between parties securely and recognizes those who are attempting to listen in on the chat (Anilkumar et al., 2024). Because quantum cryptography (QC) is more secure and safe for the future, it will soon replace traditional cryptosystems. QKD is one type of QC that guarantees the sharing of secret keys between two parties engaged in communication and detects the existence of eavesdroppers between the parties. BB84 is one such QKD protocol. The real key rate and security bound differ from the theoretical value because the implementation's devices are far from perfect. Due to this vulnerability, the hacker will take advantage of this variance to estimate the key without being discovered. Making QKD devices as near to perfect as feasible is necessary for safe key distribution. When using QKD systems in the field, proper calibration is required. Here, every potential parameter should be covered, and the BB84 source for potential information leaks caused by side channels should be analyzed. A popular implementation of the quantum communication protocol BB84 uses four laser diodes to transmit weak coherent pulses that may not have the same characteristics. These lasers have been studied for mismatch in a number of characteristics, including polarization, peak wavelength, spatial mode, spectral width, pulse width, and arrival timings at the receiver. This data is used to evaluate information exchanged between the eavesdropper and the source to determine potential leaking of information via side-channel attacks (Biswas et al., 2021; Figure 7.2).

The information-theoretic security of QKD, a secure communication technique, is based on the intrinsic unpredictability of quantum mechanics. BB84 is the earliest and most used QKD protocol; therefore, it is crucial to demonstrate its security. A variation of the decoy BB84 approach is the discrete phase-randomized BB84 procedure. When creating high-speed QKD systems, it has shown promise. An analysis with a limited number of pulses is still missing, though it provides a thorough security study of the discrete phase BB84 protocol utilizing two distinct approaches in various scenarios. To find the ideal parameter settings, the study uses simulations and optimizations. The key rate may be calculated by assuming that for a moderate number of finite pulses (107), a continuous phase randomization mechanism was in action, as long as there are more than 30 distinct phases. However, for a comparatively smaller set of discrete values, namely, 16 discrete phases, a numerical technique has been devised to determine the key rate. (Jin et al., 2024) verified that with a limited number of pulses, its performance is diminished but still manageable. In concept, QKD provides unwavering security. Numerous QKD methods have been put out and proven to provide secure communication between two users who have been verified. Compared to discrete variable (DV) QKD, CV-QKD has several benefits, including being more affordable, compatible with modern classical communication technologies, effective even in the presence of daylight, and providing a greater secure key rate. In light of this, a discrete-modulated CV-QKD system in free space is shown to be resistant to polarization drift. Additionally, the simulation results

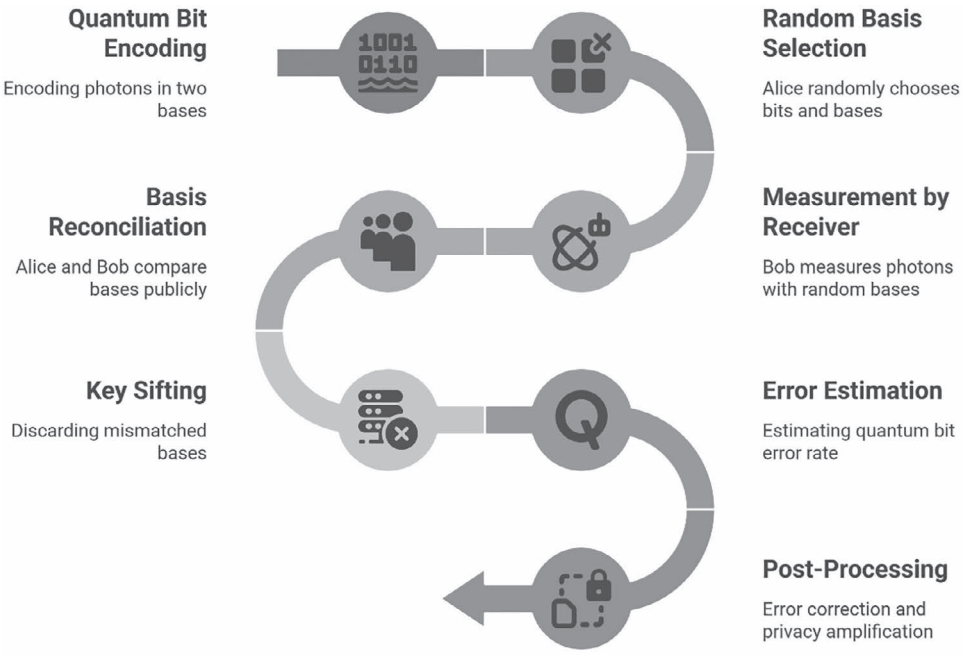


Figure 7.2 Securing communication with the BB84 protocol.

using a noise model are taken into consideration to show the impacts of different parameter modifications on the secure key rate as well as channel noise. The verification of the experimental values obtained for the implemented CV-QKD is aided by these simulation results (Rani et al., 2023). One of the most important topics in quantum information theory is QKD. There have been significant QKD systems created for about 40 years. Initially, QKD methods relied on switching between bases, specifically non-orthogonal state encoding, to provide security. The BB84 protocol is the most well-known example. Other methods for orthogonal state encoding were later created. The GV protocol and order-rearrangement protocols, two QKD methods based on orthogonal state encoding, are examples of such protocols. They are shown to be quite efficient when taking into account the use of both qubits and classical bits, with one of them not requiring the use of order-rearrangement techniques while the other does. Additionally, fewer fully entangled states may be used than in the past. Two protocols are highly efficient in terms of qubit and classical bit consumption and can be improved by reducing maximally entangled states was demonstrated (Shu, 2022).

7.3 ADVANCED PROTOCOLS AND MDI-QKD

MDI-QKD was introduced to mitigate measurement device vulnerabilities, typically the weakest link in a QKD system. In MDI-QKD, both parties transmit quantum signals to a third, possibly untrusted, measurement station. Security holds even if the station is breached, significantly shrinking the attack surface. Next-generation protocols also venture into ideas such as device-independent QKD, where no trust in the devices themselves is assumed. These techniques are critical for making the transition of QKD from laboratory to real-world, large-scale networks. MDI-QKD has been experimentally demonstrated over metropolitan

Table 7.1 Key Aspects of Measurement-Device-Independent (MDI) QKD

Protocol/category	*Description*	*Advantages*	*Challenges*
Decoy-state QKD	Uses additional (decoy) photon states to detect photon number splitting (PNS) attacks	Enhances security with imperfect photon sources—better performance over longer distances	More complex hardware setup—requires careful calibration
Measurement-device-independent (MDI) QKD	Removes trust from detection devices by measuring entangled photons in a third-party location	Immune to all detector-side-channel attacks—increases practical security	Requires precise synchronization—higher system complexity
Continuous-variable (CV) QKD	Uses modulated light fields (quadrature measurements) instead of single photons	Compatible with existing telecom infrastructure—high key rate potential	Sensitive to noise. Complex error correction needed
Twin-field (TF) QKD	Combines features of MDI and phase-based QKD to enhance distance	Breaks rate–distance limit—suitable for ultra-long-distance links	Phase stabilization is technically demanding
Quantum digital signature (QDS)	Allows for unforgeable message authentication using quantum states	Offers long-term security for authentication	Requires entanglement and quantum memory (still in development)

and even intercity links. It is a major step toward practical, robust quantum security. Key aspects of measurement-device-independent is described in Table 7.1.

7.3.1 Concept of MDI-QKD and Its Resistance to Detector Attacks

The MDI-QKD is immune to detection-side attacks, but its weak coherent photon sources can be susceptible to PNS attacks. Existing protocols use the decoy-state method, which can resist PNS attacks, but it only generates secure keys using single-photon components. To analyze weaker PNS attacks, researchers retrieve missing information from existing protocols, extract normal distribution statistics, and provide a detection method and Type I error probability. If an attack is detected, the decoy-state method is used to estimate the secure key rate. If successful, Bell state measurement events occur, secure keys can be generated using both single-photon and multiphoton pulses. Experimental results show the correctness of the method based on 5% significance level (Chen et al., 2022). For QKD, an inherently stable all-fiber polarization encoder that does not require calibration has been presented, guaranteeing stability over an extended period of time. Compared to earlier encoders that used manual polarization controls, this portable and incredibly stable encoder is a major gain. For several hours without calibration, the polarization-modulated unit (APMU), which is all fiber and calibration-free, can sustain steady encoding. The suggested APMU makes the polarization-encoding MDI-QKD more feasible by simplifying operations for Alice and Bob. Proof-of-principle is demonstrated using the four-intensity decoy-state approach (Chen et al., 2023). Device and theoretical differences pose a danger to the security of QKD. To protect against source vulnerabilities like Trojan horse assaults and state preparation errors, a reference mechanism was put forth. The security of an effective four-phase MDI-QKD employing laser pulses against possible source flaws is demonstrated in this work using the reference approach. The technique shows promise for the practical implementation of secure QKD with device defects by achieving a secure key rate of 253 bps with a 20 dB

channel loss (Gu et al., 2022). The ideal QKD protocol requires perfect random numbers for bit encoding and basis selection. However, practical quantum devices can be imperfect, affecting system security. A study reveals that weak randomness in MDI-QKD with finite resources can affect the final secret key rate, posing additional security risks. Strengthening state preparation devices is crucial for ensuring QKD system security (Jiang et al., 2022). In the terahertz frequency range, a CV-MDI-QKD protocol has been used for the first time to perform a secret key rate analysis against collective attacks. The proposed THz CV-MDI-immunity QKDs to all detector attacks significantly improve the security assurance of real-world THz CV-QKD implementations. Furthermore, the performance of practical THz CV-MDI-QKD systems is significantly influenced by finite key length (finite-size effects) and imperfect reconciliation efficiency. Increasing the modulation variance and employing a large number of keys or signals can greatly mitigate the negative effects of the finite-size effect and poor reconciliation efficiency. These discoveries are crucial to improving THz CV-QKD technology's feasibility and bringing it closer to practical applications (Liu et al., 2024a).

Among all the existing QKD methods, the reference-frame-independent MDI-QKD is unique due to its excellent security and usefulness. It can reduce the requirement for reference-frame calibrations and remove any detector-side-channel assaults. However, the performance degradation caused by finite-size effects remains a major barrier to the implementation of RFI-MDI-practical QKD. Under wider reference-frame rotation angles, an improved RFI-MDI-QKD protocol outperforms MDI-QKD by increasing key rate and transmission distance while avoiding the finite-size impact. Successful practical demonstration of the proof-of-principle at a distance of 300 kilometers (Liu et al., 2021). With CV-MDI-QKD, attacks using side channels on detectors are eliminated, whether they are known or unknown. Unbalanced modulation at transmitters affects the security of the CV-MDI-QKD system, mainly using a coherent state and a squeezed state under symmetric and asymmetric distances. However, it is challenging to fully implement assumptions in the security demonstration model, which means that potential security vulnerabilities will inevitably exist in the working system. Using an extended theoretical scheme and assuming two distinct sender modulation topologies, the critical parameter that the protocol may achieve with the unbalanced modulation mechanism is assessed. Imbalanced modulation can significantly increase secret key rate and transmission distances, demonstrating its benefits in system key parameter estimation and providing a theoretical framework for experimental implementation and optimal modulated mode (Liu et al., 2024b)

7.3.2 Security Models and Composability in MDI Protocols

A theory for the TF QKD's composable security of the sending-or-not-sending (SNS) protocol (TF-QKD). This technique for the SNS TF-QKD protocol rigorously computes the finite-key effects in QKD with error rejection using two-way classical communication (TWCC). Here, the likelihood of tagging or untagging each two-bit random group is not independent, in contrast to the typical QKD without TWCC. Imagine a virtual set of bits where each bit is independent and identical to answer this problem thoroughly. Using specific formulas, demonstrate how the failure probability somewhat alters even if the right key rate is obtained by merely applying the Chernoff constraint in the computation. Determine key rates while maintaining stringent security and boundaries, and achieve key rates significantly above the Pirandola, Laurenza, Ottaviani, and Banchi (PLOB) bound. Key rates are obtained by significantly violating the PLOB constraint with composable security. The key rate is calculated with stringent limits and security (Jiang et al., 2021). Different MDI-QKD milestones are shown in Figure 7.3.

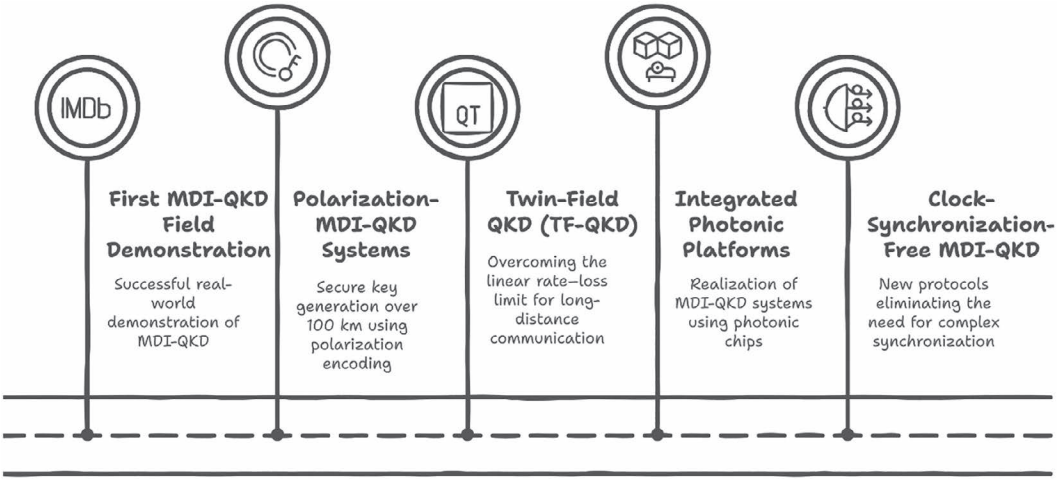


Figure 7.3 MDI quantum-key distribution milestones.

7.3.3 TF-QKD and Coherent One-Way Protocols

One of the most significant variations of the TF-QKD protocol is the SNS protocol, which has been implemented in a 511-km field fiber—the farthest field experiment to date. However, the security condition of the current TF-QKD protocols is compromised in reality since all decoy-state approaches include inevitable source faults, some of which may be non-random. By developing analogous protocols using virtual attenuation and the tagged model, this work presents a generic method for effectively determining the secure key rate of the SNS protocol with source faults. TF-QKD can improve the key rate and secure distance by studying when the source intensity is not controlled precisely. Even with a few percent intensity error, the key rate remains positive, demonstrating the protocol's robustness against source errors (Jiang et al., 2023). The well-known rate-loss restriction without quantum repeaters can be solved using the TF-QKD protocol and its variations, which have garnered a lot of interest. Weak coherent states with continuous randomized phases are often assumed in the test mode to guarantee the security of these protocols. In reality, though, it is challenging to satisfy this expectation. There are two alternative discrete-phase-randomized-TF-QKD algorithms that eliminate the phase sifting process in the coding mode to close the gap between theory and practice. Modified protocols significantly improve secret key rate in the low channel loss range, promising for practical TF-QKD compared to previous discrete-phase-randomized-TF protocols (Li & Zhang, 2023). The establishment of safe keys across great distances may be possible with QKD. The main factor impeding the point-to-point distribution of secure QKD keys distance is the channel's intrinsic loss of transmission. Increasing the QKD distance with existing technology is of major scientific interest in the pursuit of a large-scale quantum network. Authors employed the three-intensity SNS TF-QKD protocol in conjunction with the actively-odd-parity-pairing technique. Taking into account the limited size effect, the experiment shows that secure QKD across a 1002 km fiber connection is feasible. At this distance, the secure key rate is 3.11×10^{-12} per pulse. Additionally, by optimizing parameters for shorter fiber distances, performance tests on key distribution for fiber lengths ranging from 202 to 505 km were carried out. Interestingly, the safe key rate was 111.74 kbps for the typical distance of 202 kilometers between large cities (Liu et al., 2023). Challenges like scalability, synchronization, integration with classical networks, detection efficiency, setup, and cost in MDI-QKD implementation are described in Figure 7.4.

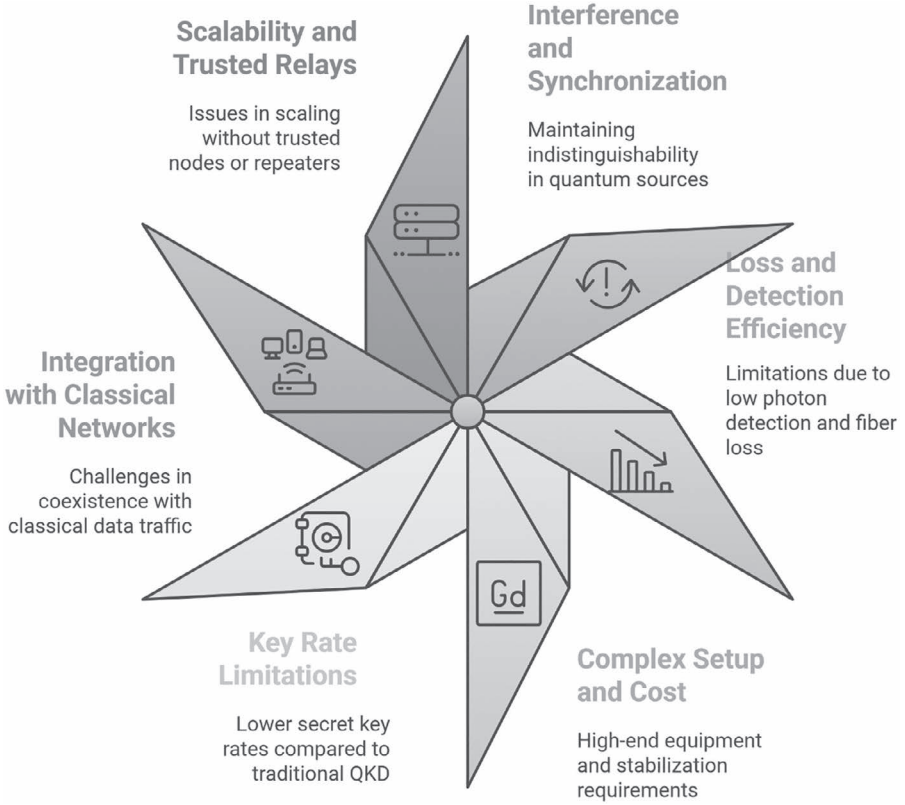


Figure 7.4 Challenges in MDI-QKD implementation.

7.4 CV-QKD AND SATELLITE-BASED QKD PROTOCOLS

CV-QKD inscribes information onto the light field quadratures instead of discrete photon states using conventional telecom equipment. This makes CV-QKD suitable for integration into existing fiber networks. In contrast, satellite-based QKD solves the problem of long-distance key distribution, exceeding the capacity of terrestrially distributed optical fibers. Missions such as China's Micius satellite have already demonstrated that QKD is possible between ground stations thousands of kilometers apart. Satellite channels make global-scale quantum communication networks possible. Both CV-QKD and satellite QKD play a central role in broadening the scope of application of quantum-secured communications beyond local settings. They are working to define the future of quantum networks. Key features of continuous-variable (CV) QKD and satellite-based QKD is described in Table 7.2.

7.4.1 Continuous-Variable QKD

One promising protocol that is simple to include in the continuous-variable nature of conventional optical communication networks is QKD. However, quantum-classical co-transmissions, such as time division multiplexing with large-power classical signals and dense wavelength division multiplexing with classical channels, make a quantum signal more susceptible to crosstalk from a classical signal. This can result in signal distortion and a reduction in key distribution performance. A carrier frequency switching (CFS)-based noise-suppression

Table 7.2 Key Features of Continuous-Variable (CV) QKD and Satellite-Based QKD

Aspect	*CV-QKD*	*Satellite-based QKD*
Basic principle	Encodes key information in quadratures (amplitude/phase) of light	Uses satellites to transmit quantum keys via free-space channels
Technology used	Homodyne/heterodyne detection, Gaussian modulation	Entangled photons or weak coherent pulses in space-to-ground links
Infrastructure compatibility	Leverages existing fiber-optic telecom networks	Requires satellite communication systems and ground stations
Advantages	High key rates—integration with classical optical systems	Global-scale key distribution—bypasses terrestrial limitations
Applications	Metropolitan QKD networks—data center interconnects	Long-distance secure communications—intercontinental QKD
Security model	Based on quantum uncertainty and Gaussian-modulated states	Secure under realistic loss/noise in free-space channels
Challenges	High sensitivity to loss and noise—complex error correction	Costly infrastructure—atmospheric loss and weather dependency
Distance range	Up to ~100 km in optical fiber (extendable with trusted nodes)	Thousands of kilometers, globally

technique that can successfully lessen the impact of large-power random noise on the weak coherent state has been developed to solve this problem. The suggested noise-suppression method shifts the transmission signal frequency spectrum to the appropriate frequency by filtering out large-power channel noise using a minimum-value window. According to simulation studies, it boosts the secret key rate by 1.43 to 2.86 times and lowers the extra noise to 1.8% (Dong et al., 2023). The CV-MDI-QKD protects measurement equipment from all side-channel assaults while enabling distant parties to exchange information-theoretic safe keys. However, the large level of untrusted equivalent excess noise in the Gaussian modulation severely restricts both the secret key rate and the secure transmission distance. More specifically, its practical implementation is directly limited by the need for homodyne detections with very high efficiency for even non-zero secure transmission lengths. By encoding key information into distinct phases of two coherent states, the CV-MDI-QKD algorithm allows for safe key distribution with low-efficiency homodyne detections while decreasing detection efficiency in asymmetric scenarios. The viability of CV-MDI-QKD employing fiber-based devices is demonstrated in a proof-of-principle experiment using a locally produced oscillator, providing a different approach for QKD with realistic security (Huang et al., 2022). With its information-theoretic security based on quantum mechanics, QKD has become a viable option for building secure communication networks. A quadratic key-rate improvement is possible with the phase-matching protocol, one of the more modern QKD techniques. Initially, symmetry-protected privacy, an abstract technique, was used to establish its security. The results of this study's re-evaluation of its security using the more logical source-replacement model are identical to those of the original proof. This model offers a novel viewpoint on the security of the protocol. A beam-splitting assault is shown as an implementation of this strategy. The resilience approach of some security analysis is further demonstrated by deriving a lower bound on the phase error rate utilizing the source-replacement model under this assault (Huang et al., 2024).

Estimating the phase noise of the CV-QKD protocol using a local oscillator, which is directly correlated with the intensity of the phase reference, is a crucial step in determining the secret key rate. However, by utilizing the transmission of the phase reference via the

insecure quantum channel, the eavesdropper (Eve) may simply launch attacks, a plan for attacking the phase reference via polarization. Only a portion of phase reference pulses are currently measured by the LLO CV-QKD system to account for polarization drift; the remaining portion is not. By adjusting the polarization direction of the unmeasured phase reference, Eve may regulate phase noise. Simulations demonstrate that, as transmission distance rises, Eve can get partial or whole key rate information. Protecting the LLO CV-QKD requires increasing the real-time phase reference intensity or the polarization measurement rate (Shao et al., 2022). Currently, only a portion of phase reference pulses are measured by the LLO CV-QKD system to account for polarization drift; the remaining portion is not. By altering the polarization direction of the unmeasured phase reference, Eve is able to manage phase noise. As the transmission distance grows, simulations demonstrate that Eve may get partial or whole key rate information. To safeguard the LLO CV-QKD, it is essential to increase the polarization measurement rate or the real-time phase reference intensity (Shelar et al., 2023).

7.4.2 Recent Developments in Satellite-to-Ground QKD

The foundation of the global quantum network is satellite-to-ground quantum communication, which signals the arrival of the quantum information age. The simplicity, stability, and convenience of implementation of CV-QKD make it a promising option for space-ground quantum communication, particularly when it comes to the resilience of space background light noise. Recently, the discrete-modulated continuous-variable protocol has gained greater attention because of its affordable implementation requirements, acceptable security key rate, and significant compatibility with current infrastructures. Through numerical simulations under different situations in free-space channel settings, it was demonstrated that satellite-to-ground communication is possible with discrete-modulated CV-QKD techniques (Li et al., 2024). Since the counterfactual QKD does not depend on any signal passing via the quantum channel, it can offer a security benefit over the conventional BB84 protocol in situations when Eve is unable to completely access the signal. In counterfactual QKD with untrusted detectors, the requirement to reveal "which detector clicked" has emerged as the primary weakness in all existing counterfactual QKD protocols, potentially compromising the security of practical implementations in untrusted-device scenarios. Two counterfactual QKD protocols—a very efficient variation and a modified Noh09 protocol—are examined for security against eavesdropping strategies that resemble the memory attack on device-independent QKD. The variation is effective against side-channel assaults and other attacks that weaken detectors, but the modified protocol is safe in situations involving untrusted detectors. Figure 7.5 illustrates how Alice separates the encoded signal into two paths, a and b. The click at D1 is the only way Alice can definitively grasp Bob's bit since there is no signal travelling over the quantum channel, another iteration of an unreliable detection device. Alice's detector is an unnamed quantum device called Fred. Security is at risk as Fred and Eve can steal information together (Lin et al., 2023).

Security against eavesdropping strategies akin to the memory attack on device-independent QKD is examined for two counterfactual QKD protocols: a modified Noh09 protocol and a more efficient variation. Side-channel attacks and other attacks that affect detectors are effectively prevented by the version, but the modified protocol is safe in situations involving untrusted detectors. The efficiency and security of the suggested framework for QKD with two non-orthogonal photon states are assessed. According to simulation findings, both the transmitter and the receiver exhibit greater percentages of original sequence recovery, error elimination, and optical spectrum enhancement. Both

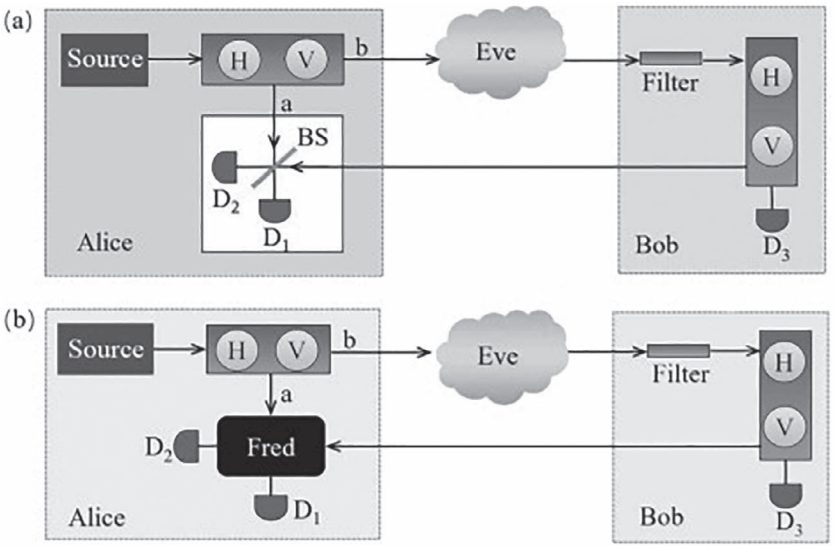


Figure 7.5 (a) The original CQKD plan. Alice divides the signal that has been encoded into two pathways, a and b. Without any signal moving over the quantum channel, Alice can only gain a definitive understanding of Bob's bit through the click at D1. (b) Version of an untrusted detecting gadget. Fred, an unidentified quantum gadget, is Alice's detector. Because Fred and Eve can jointly steal information, security is jeopardized (Lin, Y.-Q. et al., 2023).

the sender and the recipient are able to get a security key since error correction is computed every iteration (Mallick et al., 2023).

7.5 AUTHENTICATION, ERROR CORRECTION, AND PRIVACY AMPLIFICATION

In QKD, authentication is essential to ensure the identity of the communicating parties and to avoid man-in-the-middle attacks. Although QKD ensures secure transmission of keys, classical cryptographic techniques are frequently employed to authenticate the initial establishment. Error correction is required to reconcile noise and loss discrepancies in the quantum channel. Privacy amplification methods then extract a shorter, very secure key from the reconciled key by erasing any information that might be known to an eavesdropper. These operations are essential to convert raw quantum transmission into a usable, secret key. They collectively guarantee the integrity and confidentiality of quantum communication sessions. Higher protocols refine these steps to achieve maximal key rates and trustworthiness.

For QKD methods to be secure, authentication is essential, using Polynomial Hash and its variations in QKD protocols to authenticate messages of varying length. Since universal hashing is used not only for authentication in QKD but also in other QKD steps like error correction and privacy amplification, as well as in several other areas of QC, the most efficient universal hash function families, Polynomial Hash and its variants, can be used in these important steps and areas as well. Numerous variations of Polynomial Hash were examined. Every variation was shown to provide an ε-almost-Δ-universal family of hash functions. It offers a way to convert these families into an ε-almost-strongly universal family that can be used in the Wegman–Carter MAC architecture, a QKD protocol authentication

technique that is universally composable (Bibak et al., 2022). Quantum computation and quantum information are poised to revolutionize every field of science and technology as the nature-powered quantum revolution gains traction. One such area that has made significant strides both theoretically and practically is QC. One of its essential components is QKD, which necessitates the use of an information reconciliation (IR) protocol to fix any mistakes in the key bits that are conveyed across the physical system. Because of its ease of use and great error-correction efficiency, Cascade is one such protocol that is well-liked among system engineers. A practical approach to improving and implementing key optimizations in the Cascade protocol, aiming to increase efficiency without altering its foundational structure, was provided. It delves into the mathematical and physical interpretation of key parameters and quantities, utilizing the cascade of errors to minimize information leakage. A visual walk-through of Cascade, providing explanations at every step, was aiming to give developers and researchers a tool for insight-driven mistake correction (Choudhary & Wasan, 2023).

The low cost, worldwide scalability, and enhanced agility of cloud storage have drawn a lot of attention. Since unscrupulous hackers or unapproved cloud operators frequently compromise personal data or company secrets, data privacy is a crucial concern when it comes to cloud storage. A Modified Key Sifting Scheme (MKSS) is suggested as a solution to this problem to provide a secure QKD with Two-Way Identity Authentication (TWIA) for cloud storage. An MKSS that minimizes the key distribution time based on synchronization is proposed using the Tree Parity Machine (TPM). Cloud storage's low cost, worldwide scalability, and enhanced agility have drawn a lot of interest. Because unauthorized cloud operators or malevolent hackers frequently compromise personal data or company secrets, data privacy is a critical concern when it comes to cloud storage. To solve this problem, an MKSS is suggested to provide a secure QKD that combines TWIA with cloud storage. To reduce the key distribution time based on synchronization, an MKSS is proposed using the TPM. This MKSS-QKD-TWIA takes into account four distinct objects in a communication scenario: the data user, data owner, server, and data storage (Hole et al., 2023). One possible method for sharing unconditionally safe keys between distant parties is QKD. Reconciliation is a crucial component of a real-world QKD system because it corrects errors caused by sharing data across a public classical channel: quantum channel noise. A new low-density parity-check method based on a syndrome reconciliation protocol minimizes information leakage by making full use of previously wasted syndrome information. The results of simulations and theoretical research demonstrate that our protocol may clearly lower the number of communication rounds and information leakage (Mao et al., 2021). In QKD, two users utilize a quantum communication channel to extract a shared secret key while an eavesdropper is present. Because they can produce large key rates from the arrival times of entangled photons, QKD techniques based on energy-time (ET) entanglement of photons have been the subject of much research. In previous work, (Zhou et al., 2023) suggested a system termed multi-level coding (MLC) for the IR stage of ET-QKD protocols, where users interact over a classical channel to reconcile disparities in their data. The MLC scheme is a low-complexity decoding method for IR in ET-QKD systems, but it has poor error-correcting performance and does not fully utilize the QKD channel's properties. Non-binary MLC, or NB-MLC(a), is a versatile technique that utilizes a positive integer as its parameter. With little increase in complexity, this procedure greatly increases key rates. Key rates are further improved using an interleaved decoding and communication (IDC) method and a method for joint rate and degree distribution optimization. With the inclusion of QKD channel information, the JRDO algorithm achieves a greater key rate than earlier codes. In addition, the IDC technique enhances crucial rates over earlier approaches. In general, the NB-MLC(a) approach exhibits a noteworthy 40–60% enhancement of the critical rates when compared to earlier research (Mitra et al., 2024).

7.6 NETWORK ARCHITECTURES AND ROUTING PROTOCOLS FOR QKD

Creating QKD networks means developing architectures that effectively disseminate keys over multiple users and locations. Trusted-node networks, where keys are passed through intermediary stations, are a prevalent solution today. Quantum repeaters, although still experimental, hold out the promise to stretch distances without relying on intermediate nodes. QKD routing protocols need to take into consideration quantum-specific limitations. Hybrid quantum-classical networks are being considered for combining QKD with the existing internet infrastructure. The Tokyo QKD network is an example of the experiment's feasibility at a large scale, as well as the European Quantum Communication Infrastructure (EuroQCI). Successful routing protocols are essential to make QKD scalable and feasible.

7.6.1 Architectures of QKD Networks: Point-to-Point, Trusted Node, and Quantum Repeater-Based

One of the key challenges in multi-hop QKD networks is achieving secure and rapid packet routing. Examine a realistic trusted-node scenario in which a QKD protocol creates symmetric private key pairs at random across any network link that supports QKD. Prior to being sent point-to-point, packets are first encrypted using the available quantum keys. The design of a safe and capacity-achieving routing strategy that accounts for the different physical-layer link capabilities and the time-varying availability of the encryption keys is a critical issue in this context. Tandem queue decomposition (TQD) is a novel, secure throughput-optimal technique designed to solve this issue. Secure capacity for multiple traffic classes is achieved via the TQD policy, which integrates the QKD process into the Universal Max Weight routing policy. It eliminates crucial availability limitations and simplifies the issue to a generic network flow problem. Lyapunov stability theory is used to establish the throughput-optimality of the policy, and its performance is quantitatively compared to that of other routing algorithms using a simulator (Akhtar et al., 2023). The QKD has emerged as one of the most important contenders for future security features as it guarantees absolutely information-theoretic secure communication that is guaranteed by the rules of physics. To increase the benefits of QKD protocols for numerous users, a large-scale network with a scalable and integrated scheme is needed. This is a proposal for a 16-user fully linked quantum network that uses a new time-bin entangled source that is incorporated into multi-channel periodically poled lithium niobate waveguides. The quantum processor uses dense wavelength division multiplexing technology to produce 120 entangled photon pairs based on this entangled source, allocating 15 connections to each user. For user-to-user communication, a phase-compensated Mach-Zehnder Interferometer is created. Sixteen users can communicate across large distances using the network. The new time-bin entangled source approach opens up new possibilities for large-scale and extendable quantum networks with integrated photonic architecture by enabling the effective deployment of large-scale quantum resources in a small platform (Huang et al., 2025).

7.6.2 Role of Hardware, Photonic Components, and Inter-Network Synchronization

One of the most popular implementations of QKD, which uses quantum physics to offer secure communication, is the BB84 protocol. However, BB84's efficiency and scalability are limited by the traditional post-processing stages, such as filtering, mistake correction, and key verification, which frequently result in a large communication overhead. PRNG-based predetermined key bit positioning, which removes unnecessary bit exchanges during sifting, hash-based

subsequence comparison, which makes key verification lightweight and effective, and adaptive basis reconciliation, which reduces the communication costs related to basis matching, are the three main optimizations for BB84. A detailed performance analysis demonstrates that, in comparison to traditional QKD techniques, the proposed improvements save communication costs by 50% for large key sizes. Other DV-QKD protocols, such as six-state, B92, and E91, can benefit from the BB84 protocol's improvements. The broader use of quantum communication systems is supported by these enhancements, which increase post-processing efficiency and scalability and make them appropriate for bandwidth-constrained applications such as IoT networks, secure banking systems, and defense communications (Bhatia et al., 2025).

Data security and traditional cryptography may be threatened by the impending age of quantum computers. With many protocols, QKD offers unwavering security across vast distances in fiber and free space communication. The unique features of superconducting nanowire single-photon detectors (SNSPDs), including low dark counts, picosecond temporal resolution, and near-unity efficiency in the infrared, are making them a dominating technology for QKD. NbN SNSPDs at 2.2 K using a CW laser source at 1550 nm, proving their high efficiency at low dark counts with a high counting rate, consistent with QKD requirements for long distances or high secure key rates was characterized (Bruscino et al., 2024).

The wireless sensor network (WSN) is a dense network of sensors that processes and sends data to a sink node, dependent on environmental conditions. Wireless body sensor network (WBSN) technology is one of the main areas of growth in the entertainment and healthcare industries. WBSNs are a preferred choice in these sorts of circumstances due to their small size, low cost, low power consumption, little maintenance needs, and simplicity of installation. WBSNs have numerous benefits, but they also have some drawbacks, such as limited power supply and network mobility, which affect their ability to communicate, store data, and process information. The Improved Unequal Secure Cluster-Based Distributed Routing Protocol with quantum-key distribution (IUSCDRP-QKD) is a hybrid solution for WSNs to improve energy efficiency, reliability, and security. It allows direct and confidential communication between sources and destinations using a single-photon exchange. QC safeguards the distribution of symmetric encryption keys, providing robust and unbreakable security. The goal is to increase the protocol's efficiency percentage to the maximum (Muthusamy et al., 2024).

In the current situation, energy-efficient protocols for mobile wireless smart dust networks are becoming more and more popular, outlining a unique intrusion detection framework approach that is better suited for periodic data collection in secured clustered mobile smart dust networks. Improving the transmission energy is the primary goal of mobile wireless smart dust nodes that are battery-operated. The suggested approach has the advantage of lowering computational and communication overheads and providing an enhanced energy-efficient routing protocol in terms of intrusion detection and energy efficiency. Complex security protection algorithms consume more energy, yet intrusion and disruption are hard to detect. Algorithms for specific encryption and IDS-based clustering aid in detecting assaults on mobile nodes and lowering energy consumption. Energy use is influenced by the pace of data transmission. Improved energy-efficient routing algorithms for mobile wireless smart dust networks are demonstrated via simulations in NS2 (Rajesh & Rajanna, 2022).

7.7 EMERGING TRENDS AND FUTURE DIRECTIONS IN QKD

New trends in QKD are device-independent QKD, photonic chip integration for compact QKD systems, and quantum repeaters for long-distance communication. Efforts are being made to increase secret key rates, lower hardware expenses, and create standardized protocols. There is increasing demand for the integration of QKD with post-QC to

enable layered security solutions. In addition, quantum internet projects try to construct networked quantum computers and sensors through QKD-secured connections. Artificial intelligence is also being explored to optimize fault detection and network management for QKD. The future of QKD will have a key role to play in securing communications in a quantum-enabled world.

Recent advancements in quantum computing have put the conventional security system, which relies on symmetric and asymmetric key exchange protocols, in jeopardy. A novel QKD protocol was introduced to produce the secure key without keeping the qubit by the sender or receiver, trustworthy parties. In any of the following locations: {(1,2,3,4), (1,3,2,4), and (1,4,2,3)}, a pair of bell states is produced by the coupling of four random classical bits. Some study uses "$n/2$" groups to generate a raw key, while the remaining groups check for Eve. The flying bell states scatter due to Eve's incorrect measurement basis, causing entanglement swapping. The sender adjusts the key based on the receiver's position preference. The intercept-measure-resend attack evaluates the trade-off between key generating rate and disturbance. Circuit simulation is demonstrated in IBM Quantum Lab (Babu & Jayaraman, 2023). Information-theoretic security is provided via quantum conference key agreement (CKA), which permits key exchange between several trustworthy users. At the moment, the majority of quantum CKA protocols' key rates are hampered by the maximum efficiency of all quantum channels. A quantum CKA protocol with three users was developed, drawing inspiration from the coherent one-way and TF-QKD protocols. The restriction can be broken by our approach by using coherent states of intensity 0 and μ to encode logic bits. Furthermore, our methodology eliminates the need for repeated intensity modulation and phase randomization, which simplifies its experimental demonstration (Cao et al., 2021). With its intrinsic security properties that have been substantiated, QKD has emerged as a revolutionary technology. Binary encoding-based qubit-based QKD techniques are inherently limited by the size of the secret key. The maximum capacity of the secret key is limited to one bit per photon due to this constraint. However, because qudit-based QKD techniques allow for the transmission of several secret bits per photon, they are advantageous in situations when photons are limited and noise is present. The present constraint is the maximum distribution distance, which stays at 20 km fiber distance, despite the fact that proof-of-principle entangled-based qudit QKD systems have been successfully proven over the years. In a 50 km optical fiber link, a high-dimensional time-bin quantum steering algorithm based on ET entanglement produces cryptographic keys and safe finite-length keys at 0.24 Mbits s–1. The nonlocal dispersion cancellation approach is used to secure the system, which is constructed with commercial components. It establishes new benchmarks for quantum steering and ET entanglement observations at various fiber lengths. When it comes to secure key throughput utilizing commercial single-photon detectors (SPDs), the protocol performs better than qubit-based QKD techniques (Chang et al., 2024).

It has been suggested to use the source-type PLS method, which uses the turbulent channel as the source of randomness, only at the initialization stage, to address a number of issues with QKD, post-quantum cryptography (PQC), and PLS protocols. It has been suggested to modify the protocol's parameters based on time-varying FSO channel circumstances in light of the low raw key rates for source-type secret key generation protocols, which are determined by the lengthy coherence time of atmospheric turbulence channels. PQC, computational security-based cryptography, and QKD all have constraints that must be addressed. It is suggested to use atmospheric turbulence as a shared source of randomness in a cryptosystem that is controlled by it. To get around low secret key rates, the technique modifies the gamma-gamma distribution parameters every Tc seconds. Compared to conventional QKD and source-type physical-layer security techniques, the secret key rates are

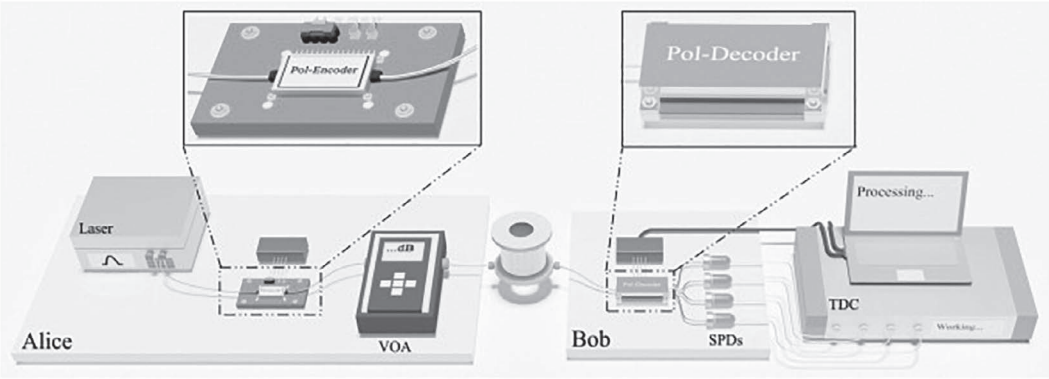


Figure 7.6 Experimental setup of the silicon-based integrated QKD (Du et al., 2023).

substantially greater, and the atmospheric turbulence channel shapes the final key (Djordjevic, 2021). To achieve a finite-key secret rate of 240 bps across a 100 km fiber spool and to automatically compensate for fiber polarization drift, a totally chip-based decoder for polarization-encoding QKD was developed. A typical silicon photonics foundry was used to create the device, which had an average quantum bit error rate of 0.59% in the absence of polarization feedback. This work represents a major advancement toward the large-scale, integrated, and useful implementation of QKD systems (Du et al., 2023).

Figure 7.6 depicts Alice creating pulsed light with a 50 MHz repetition rate using a laser. For the purpose of randomly modulating the decoy and polarization states, the produced pulsed light is linked to an encoder chip that has both a polarization modulator and an intensity modulator. The off-chip VOA then attenuates it to a single-photon level before sending it to Bob over a fiber spool. The time-to-digital converter records the detector data after Bob uses a decoder chip to decode the incoming photons and four SPDs to detect them (TDC). A personal computer is used to process the detector events that the TDC has captured. A polarization feedback loop connects the decoder chip's electronics module to the personal computer. To encrypt and decode communications using symmetric cryptosystems, two parties can generate and exchange a random secret key using key distribution. Thus, particularly in wireless networks, this is frequently regarded as the most basic cryptographic primitive of secret communications. The QKD, which incorporates quantum mechanical elements and may be regarded as absolutely safe, differs from the conventional approach, which is predicated on the notion that certain mathematical problems are hard (Escanez-Exposito et al., 2023). To facilitate the sharing of a secret key between a quantum participant and a classical participant, (Wang et al., 2011) presented a single-state semi-QKD mechanism. To get portions of the final shared key without the parties noticing, an eavesdropper can employ a double C-NOT attack (Gu & Hwang, 2022). The key rate of the TF-QKD SNS protocol may surpass the linear bound of repeaterless QKD, and it can tolerate substantial misalignment errors. Perfect vacuum sources are necessary for the original SNS protocol and all TF-QKD variations; however, there are no workable perfect vacuum sources in real-world tests. Instead, experimenters replace vacuum sources with really weak sources, potentially breaching the protocol's security. Here, non-asymptotic decoy-state analysis of an SNS protocol with defective vacuum sources was provided. The numerical simulation reveals that our protocol can achieve a similar key rate with perfect vacuum sources, thereby closing the potential security loophole in TF-QKD (Hu et al., 2022)

Bi-directional quantum communication is used by two-way QKD protocols to create a shared secret key. Security evaluations are still difficult because of the larger attack surface. Here, an information-theoretic security study in the finite-key context is carried out, and a high-dimensional version of the Ping Pong protocol is examined. A new proof methodology was studied for two-way QKD protocols, based on Bouman and Fehr's quantum sampling framework and 2019 sampling-based entropic uncertainty relations, and highlights the benefits of high-dimensional quantum states in two-way quantum communication (Krawec, 2022). For the first time, repeated code is used to develop a novel QKD technique. Using settings from a threshold protocol, it converts the quantum state sequence and encodes the classical key sequence. This protocol is more effective and economical since it verifies participant identities, thwarts assaults, and conserves quantum resources (Li & Li, 2022). A flexible transmitter that may execute QKD algorithms with discrete and continuous variables (DV-QKD and CV-QKD, respectively) was studied to implement a GG02 CV-QKD protocol with a genuine local oscillator over a 10.5 km channel and a time-bin-encoded BB84 DV-QKD protocol over a 47 km physical quantum channel using this transmitter. The secret key rates for DV-QKD and CV-QKD are 4.1 kbps and 1 Mbps, respectively. For re-configurable optical networks, where the QKD protocol is chosen to maximize performance based on connection characteristics, the presented transmitter method is especially appropriate (Lopez Grande et al., 2021). By taking into account the one-decoy approach and examining its performance under the impact of a detector after pulse, the tight key analysis for the SARG04 protocol can provide more resilience against PNS assaults than the BB84 protocol, which is implemented using weak pulses. Our findings show that the secure key rate and the maximum transmission distance both slightly increase with block size. Crucially, in real-world applications, the detector after pulse is essential and affects the SARG04 protocol more strongly than the BB84 protocol (Nian et al., 2024).

7.8 CONCLUSION

QKD represents a revolutionary leap in secure communication, providing provably secure communication founded upon the inherent principles of quantum mechanics. Backbones like BB84, in addition to breakthroughs such as decoy-state, TF, and MDI protocols, have propelled substantial evolution in surpassing attacks like PNS and side-channel attacks. Enhancements at the system level, such as polarization-modulated MDI-QKD systems and improved TF-QKD schemes, have also increased performance and security over long-distance transmission. Recent advances have also brought QKD's utility to vital infrastructures such as hydropower and nuclear power plants, enhancing the resilience of cyber-physical systems. Increasing the feasibility of CV-QKD and satellite-based QKD heralds the establishment of global-scale secure communication networks. Concurrent developments in authentication, error correction, and privacy amplification—specifically the application of sophisticated reconciliation protocols based on LDPC codes—have enhanced overall efficiency and reliability.

At the network level, QKD integration is progressing with advances in quantum routing algorithms and cluster-based communication protocols, the foundation for achieving scalability and flexibility for future quantum internet architectures. As QKD transitions from laboratory demonstrations to deployment in the real world, additional innovations in hybrid technologies, detector performance, and resource optimization will continue to fuel the next wave of breakthroughs. QKD has come a long way from theoretical ideas to applied technologies that address contemporary cybersecurity requirements. While there are still issues in standardization, cost minimization, and mass deployment, the direction is clear: QC will be a pillar of tomorrow's secure digital infrastructure.

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Chapter 8

Exploring the Evolution of Quantum Teleportation Research in Quantum Cryptography

A Scientometric Evaluation

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8.1 INTRODUCTION

Cryptography involves the technique of safeguarding communication by transforming plain text into ciphertext. It enables users to communicate securely over an insecure communication network and also ensures that the data is only available to the intended recipient, preventing access by any unauthorized parties [1]. In the realm of the digital world, classical cryptography has successfully performed this role over time, but, with the advent of quantum computing, there are growing concerns about the vulnerability of encrypted information to cyberattacks. As a result, it is believed that the current traditional encryption techniques might soon be insufficient. To address the inherent vulnerabilities of classical cryptographic systems, quantum cryptography has emerged as a promising paradigm, offering enhanced security based on the principles of quantum mechanics. Quantum cryptography leverages the principles of quantum mechanics to ensure data security [2]. Qubits are the fundamental element of quantum cryptography. Unlike classical bits, which are either 0 or 1, qubits exploit superposition to allow existing in multiple states simultaneously, such as 0, 1, or both. This allows quantum computers to process such a vast amount of information in parallel. Quantum teleportation (QT) enables the secure transmission of quantum information between distant locations without physically moving the information's carrier, like atoms or photons, and plays an important role in quantum cryptography, as shown in Figure 8.1. QT, unlike the fictional representations where objects disappear from one location and reappear in another, does not involve the physical relocation of objects. Rather, it uses quantum entanglement in conjunction with classical communication to relay the quantum state of an object over vast distances 1, making it extremely difficult to intercept information, as any unforeseen access to the information would alter the quantum state, making such attempts easy to detect [3, 4]. Quantum transmission conveys solely information, devoid of the physical particles that transport it. The basis of QT is a set of quantum measurements and operations, including those described by Lee et al. [5], which include reversing superposition states to their initial configuration.

QT relies fundamentally on the phenomenon of quantum entanglement, which enables the transfer of quantum states between distant particles. Entanglement is one form of the unique correlation in quantum systems shared among particles, which are referred to as qubits. These qubits continuously update the information pertaining to their quantum state, which enables them to establish a quantum communication network. Quantum superimposed particles are entangled for the purpose of reinforcing the security and reliability of the communication networks. This method exploits the fundamental properties of quantum entanglement. One prominent use of such techniques is in the entanglement-based quantum key distribution (QKD), where pairs of entangled particles are shared during the process of forming secure cryptographic keys [6, 7]. In addition, QT follows the no-cloning theorem [2], which states that an unknown quantum state cannot be created, due to which attempts made

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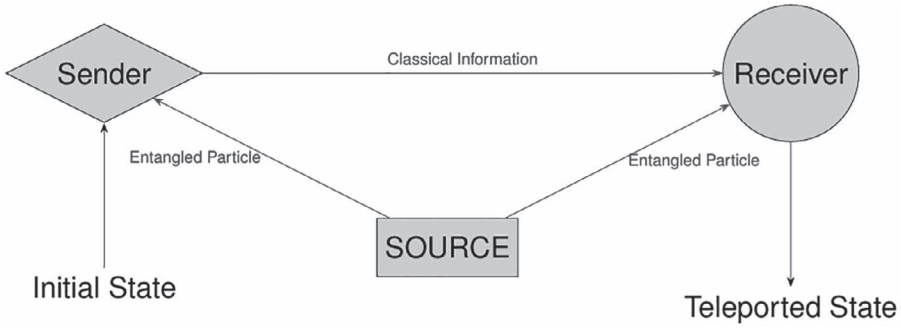


Figure 8.1 Quantum teleportation.

at intercepting the information being transmitted will lead to the destruction of that quantum state and notify authentic users about a system breach [8]. QT also plays an important role in quantum digital signatures [9]. Moreover, classical cryptography faces problems in the secure distribution of keys at long distances due to signal degradation, while QT, when used in conjunction with satellite-based networks and quantum repeaters, enables global secure communication [10, 11].

From the previous discussion, it is clear that QT plays an important role in quantum cryptography. Therefore, to systematically evaluate the research progress of QT in quantum cryptography, a scientometric analysis of articles from the Scopus database has been conducted. Scientometric analysis is used to evaluate the research trends and interdisciplinary growth. By quantifying bibliometric data, the most productive authors, countries, and journals in the field are revealed. Also, with the help of citation analysis, the core documents forming the framework of the field are obtained.

Organization of the Chapter

The rest of the chapter is organized as: Section 8.2 presents the detailed literature review of relevant scientometric studies. Section 8.3 introduces research questions to tackle our core research goals and interpret outcomes effectively. In Section 8.4, the research methodology adopted for the study is outlined. Section 8.5 delves into the results. And, finally, section 8.6 concludes the chapter with future scope.

8.2 LITERATURE REVIEW

Scientometric is a generic approach for detecting and visualizing emerging trends in scientific literature, focusing on the identification of research fronts and intellectual bases through citation analysis. Scientometric analysis helps in revealing relationships among documents and understanding the structure and evolution of knowledge areas [12, 13]. Klavans and Boyack [14] study focused on improving how relatedness was measured and directly supported creating more accurate scientometric maps of scientific knowledge. In this study, authors proposed a comprehensive evaluation framework incorporating accuracy, coverage, scalability, and robustness.

To date, no research had specifically conducted a scientometric analysis of QT in the context of quantum cryptography. However, there are studies that focus on the scientometric analysis of quantum cryptography. For instance, Olijnyk [15] presented an analysis of quantum cryptography in China, offering both qualitative and quantitative evaluations of QC research frameworks. The study showed that quantum cryptography had received heightened attention over the past decade, with China supplanting all other nations as the leading producer of quantum cryptography-related scholarly output from 2001 to 2017.

Gupta et al. [16] provided an extensive scientometric study of 10801 publications from 1992 to 2019 and calculated an annual average growth rate of 24.76% in quantum cryptography research. This boom was attributed to multiple countries, notably the USA and China, which are central to the publication. The results indicated a decline in the average number of citations per paper over time, dropping from 64.84 in the initial years to 14.06 more recently. This trend reflects a change in research influence and possibly points toward the maturation of the field. This study highlighted that key research areas include QKD and data security. Also, this study aids in maximizing future research efforts.

Additionally, Sharma et al. [17] presented a scientometric investigation of 1395 articles and analyze the contribution of material science in quantum cryptography. The study results highlighted the advanced contributions of material science to technology, identifying China, the USA, and India as the top three contributing countries. The study identified significant trends and future directions in quantum cryptography and provides insights for researchers and policymakers in the field of quantum cryptography.

8.3 RESEARCH QUESTIONS

To ensure clarity and direct alignment with the study's objectives, the findings are organized in a question-and-answer format. This structure allows for a focused examination of each key aspect of the analysis and facilitates effective interpretation of the results. The research questions guiding this scientometric investigation are as follows:

- What are the temporal trends, and how do various subjects contribute to the research productivity of QT within the realm of quantum cryptography?
- Which documents form the core intellectual framework of this domain as evidenced by citation networks?
- Which authors, countries, and journals are top contributors to the research productivity of QT in the realm of quantum cryptography?
- What are the emerging topics for QT within the realm of quantum cryptography on the basis of keyword analysis?

8.4 RESEARCH METHODOLOGY

The research methodology chosen for this analysis consists of different stages, as shown in Figure 8.2. Different stages for this study are outlined here:

Stage 1: Data Search Strategy

To obtain data, the Scopus database was utilized of its wide range of data availability and user-friendly interface. The data was retrieved using the search query employing keywords "quantum cryptography" AND "quantum teleportation" and time span: 2000–2024. Thus, we get $n = 413$ documents.

Stage 2: Screening

In stage 2, obtained documents were further filtered by defining Document type: Article and Language: English and excluding the rest of the 185 documents.

Stage 3: Data Extraction

On filtering the data, we get $n = 228$ articles, which were exported in .csv (comma-separated values) file format, which includes a variety of details like citation information, bibliography information, and abstract and keywords.

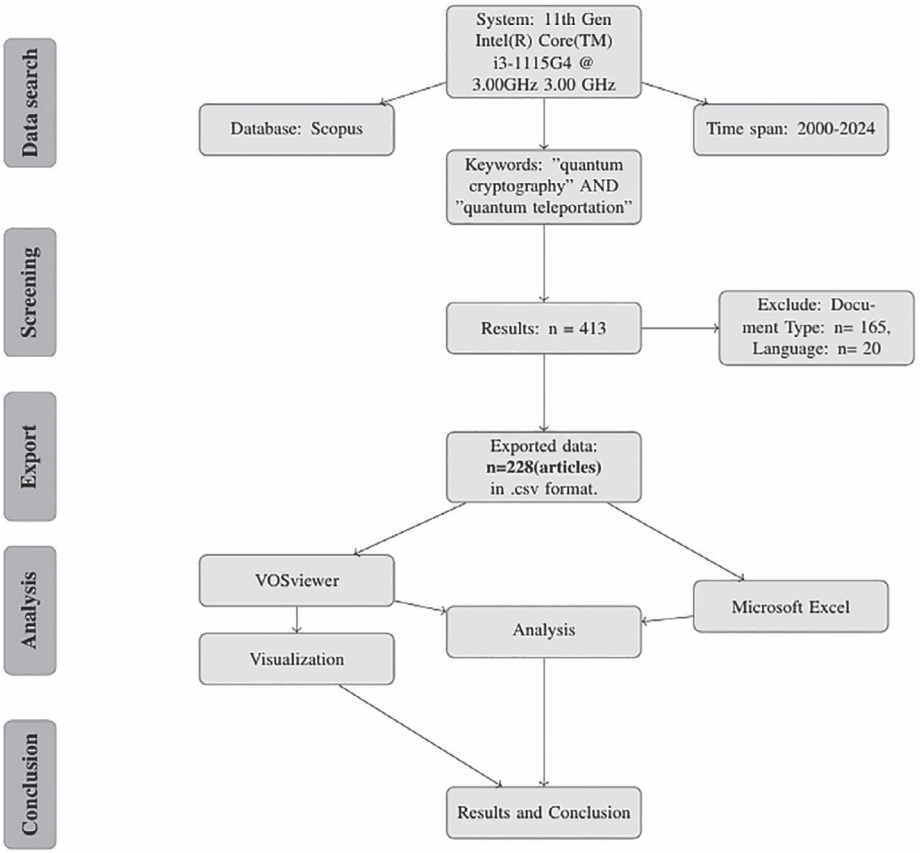


Figure 8.2 Flowchart for research methodology.

Stage 4: Analysis

For network-based analyses, VOSviewer was utilized, and citation networks and keyword overlay were created. Microsoft Excel is utilized to perform a quantitative analysis of the data. The citation counts, the productivity of countries, and so on are obtained using Excel software.

Stage 5: Results and Conclusion

In stage 5, we align findings with the research questions and obtain the answers to these questions. Further conclusions were drawn based on the obtained results.

8.5 RESULTS

8.5.1 What Are the Temporal Trends, and How Do Various Subjects Contribute to the Research Productivity of QT within the Realm of Quantum Cryptography?

To answer this question, we have analyzed the 228 articles between 2000 and 2024. Figure 8.3 presents a temporal trend in publications from 2000 to 2024. Initially, during the early 2000s, the research output was quite low since the number of documents was under 10 per year. However, from 2014 onward, an apparent increase becomes evident, which

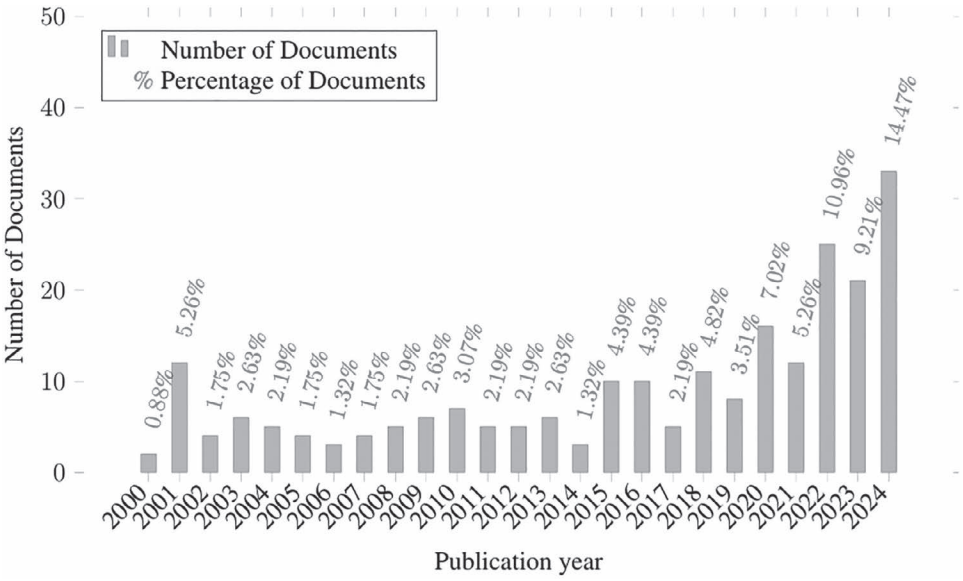


Figure 8.3 Publication trend in worldwide research productivity on QT in quantum cryptography.

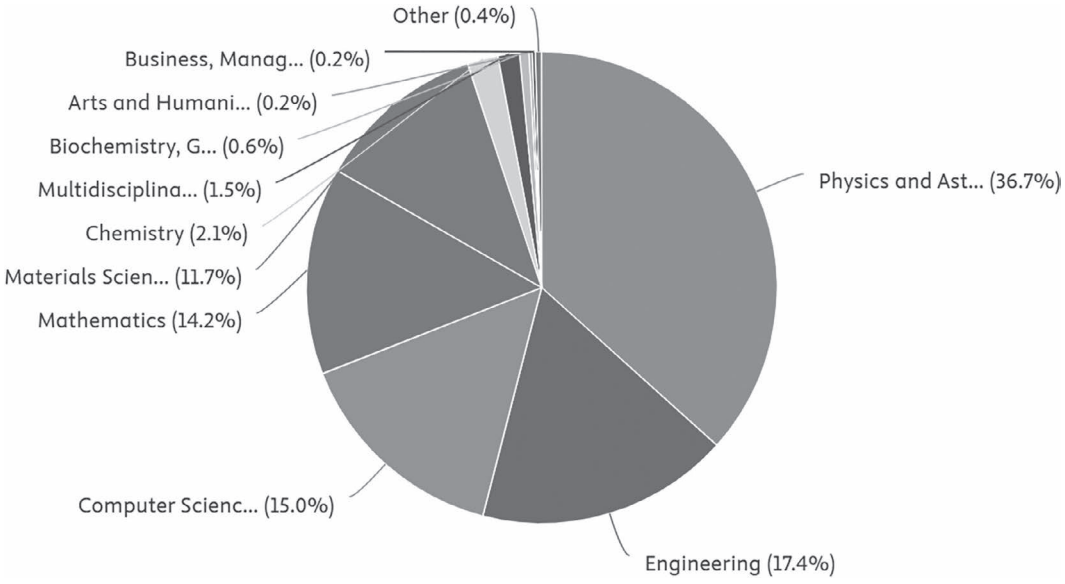


Figure 8.4 Subjectwise distribution of articles.

accelerates markedly in 2018. Publications peaked in 2024, accounting for approximately 11.4% (or 228 documents) of the total publications considered in this study. This development, most pronounced in recent years, appears to suggest an increase in academic interest toward QT, probably due to the evolving fields of quantum computing as well as the growing importance of secure communication frameworks.

The distribution of articles by subject, shown in Figure 8.4, indicates that the domain is largely anchored in Physics and Astronomy, which alone comprises 173 articles (36.7%

of the total), demonstrating its underpinning in the theoretical and experimental studies of QT. Other notable contributions come from Engineering (82 articles, 17.4%), Computer Science (71 articles, 15.0%), Mathematics (67 articles, 14.2%), and Materials Science (55 articles, 11.7%) that demonstrate the interdisciplinary expansion of the literature in relation to the computational modeling, algorithm design, mathematical formalism, and materials required for quantum devices. The small but significant participation from Chemistry, Biochemistry, Arts and Humanities, and Business Management indicates new emerging technological applications and ethical issues. To sum up, the dynamics of the research conducted alongside the segmentation of the temporal and subject focus reveal that the study of QT in the context of quantum cryptography is increasing at a remarkable pace as well as broadening in scope across fields of study, thus indicating its significance and prospects of development in the context of advanced secure quantum information processing.

Subject Area	*Count*
Physics and Astronomy	173
Engineering	82
Computer Science	71
Mathematics	67
Materials Science	55
Chemistry	10
Multidisciplinary	7
Biochemistry, Genetics, and Molecular Biology	3
Arts and Humanities	1
Business and Management	1
Other	2

8.5.2 Which Documents Form the Core Intellectual Framework of This Domain as Evidenced by Citation Networks?

This section delves into the key documents forming the core framework of the QT within the realm of quantum cryptography. To answer this question, we have analyzed the 228 documents using a document citation network. In this network 5, each node represents a particular document, and the size of the nodes varies according to the citations that document has received. The link between two nodes represents the citation relationship between the documents. Among 228 articles, the top 10 most-cited documents' contributions are presented in Table 8.1. The analysis reveals that Horodecki's et al. (2009) [18] article had received the highest number of citations, that is, 7927, and in this article the theoretical understanding of quantum entanglement is explained, which helps in advancement in quantum cryptography. Following this is Weedbrook et al. (2012) [19] with 2752 citations, in which the theoretical and practical sides of quantum information technology based on Gaussian states are explored. Further, Pan et al. (2001) [20] and Reid et al. (2009) [21] explained the theoretical understanding of quantum entanglement, which is crucial for QT.

In Ursin et al. (2007) [22], Bombin and Martin-Delgado (2006) [23], Marcikic et al. (2003) [24], and Kim et al. (2012) [25] long-distance entanglement, topological quantum codes, and long-distance teleportation had been successfully demonstrated.

8.5.3 Which Authors, Countries, and Journals Are Top Contributors to the Research Productivity of QT in the Realm of Quantum Cryptography?

8.5.3.1 Most Productive Authors

Table 8.2 presents the top 10 most productive authors and each author's document references. Among all the authors, Xiaojun Wen has the most documents, with 6 having a total of 57 citations, which relates to 9.5 citations per publication in total. Out of the total six documents of Xiaojun Wen, "Wen2008realizable" [29] stands out as the major document that employs the quantum one-way function for realizing signatures and verifications in digital multi-signature schemes, while QT is employed in the quantum multi-signature scheme. Accounting for the second spot, Ying Guo comes in fairly close with five publications and 57 total citations, suggesting a similar impact on the scholarly community. Despite having four publications, Pirandola has the most impact due to his total citation count, which sits at 2905. Also, most of his work was published in 2016. Major works of Pirandola include Weedbrook et al. (2012) [19], which is also among the top 10 contributing documents, as shown in Figure 8.5. Others include Fei et al. (2008) [30], who had published four and three articles but have been cited greatly more at 184 and 204, respectively. This indicates that they possess a far stronger impact per article. Among the four documents of Fei et al. (2008) [30], "Fei2008teleportation" [30] is the highly cited and impactful document in which the quantum secure direct communication protocol with a random basis and order is analyzed, and an effective teleportation attack is presented. Along with them, Yan Chang, Jian Li, Yongzhi Chen, and Junbin Fang are also scholars who apparently have many lesser-cited works, which suggests higher works with lesser impact.

8.5.3.2 Most Productive Countries

Figure 8.6 displays the international research productivity and the world's interest hotspots, like North America, Western Europe, and parts of Asia, which are also heavily highlighted on the map. The green shading directly correlates to document count and shows China's dominant role while also highlighting the contributions of other nations. The detailed country-wise

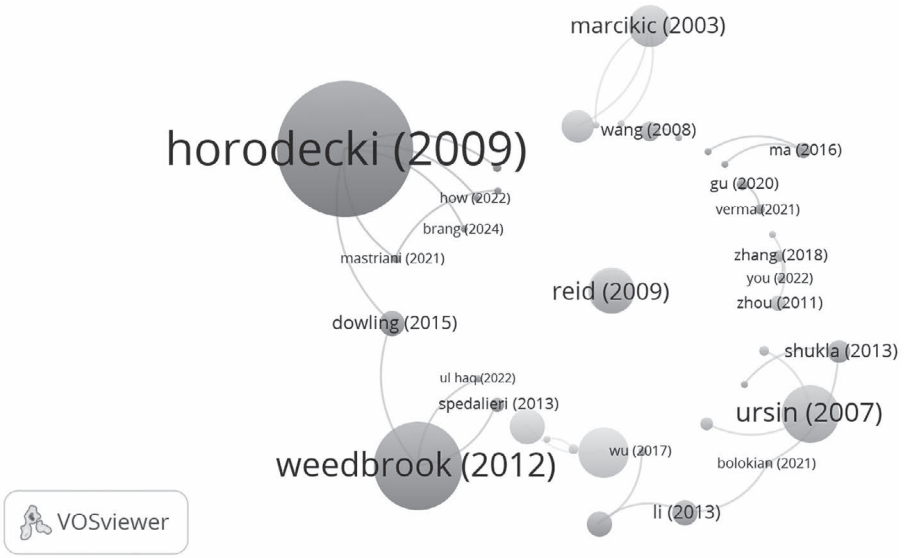


Figure 8.5 Document citation network.

Table 8.1 Contributions Made by Top Ten Documents

Sr. No.	*Document*	*Citations*	*Aim*	*Findings*	*Relevance*
1	Horodecki et al. (2006) [26]	7927	To examine the basic features of quantum entanglement	Informs how measures of entanglement, such as entanglement of formation and logarithmic negativity, are dissimilar, highlighting some measures are easily computable while others require deeper evaluative analysis	Explains the theoretical understanding of entanglement, which helps in understanding quantum mechanics, vital for developing quantum technologies.
2	Weedbrook et al. (2012) [19]	2752	To explore the theoretical and practical sides of quantum information technology centered on Gaussian states	Discusses numerous significant discoveries concerning the involvement of Gaussian states and operations in quantum information processing	Emphasizes that Gaussian quantum information processing permits an ease in the mathematical treatment of quantum protocols.
3	Pan et al. (2001) [20]	949	To explore feasible methods for entanglement purification	Proposes protocols that enhance the effectiveness and ease of distributing entangled states	Efforts to effectively distribute entangled states are crucial for developing secure communication systems.
4	Ursin et al. (2007) [22]	687	To experimentally demonstrate long-distance entanglement-based QKD over 144 km	Successfully established a quantum communication link between La Palma and Tenerife, where one photon was measured locally, and the other transmitted over an optical free-space link	The ability to maintain secure quantum communication over long distances is crucial for secure data transmission applications.
5	Reid et al. (2009) [21]	565	To analyze certain phenomena in quantum mechanics, focusing on foundational issues or applications of quantum theory	Provides additional measures or insights into quantum entanglement, a salient feature of quantum systems	Relevant for theoretical and practical measures in quantum computing and quantum cryptography.
6	Bombin and Martin-Delgado (2006) [23]	491	To construct quantum codes of a given topology that allow quantum entanglement distillation	Successfully devises topological quantum codes enabling entanglement distillation, broadening the uses of topological quantum codes	Aids in developing quantum systems that are more robust against errors through topological methods.
7	Marcikic et al. (2003) [24]	452	To demonstrate the feasibility of long-distance teleportation of qubits using telecommunication wavelengths	Successfully demonstrates long-distance teleportation of qubits, integrating with optical fiber communication infrastructure	Use of telecommunication wavelengths makes it easier to apply advancements to current technology.

(*Continued*)

Table 8.1 (Continued)

Sr. No.	*Document*	*Citations*	*Aim*	*Findings*	*Relevance*
8	Kim et al. (2012) [25]	432	To propose and demonstrate a method to protect entanglement from decoherence using weak measurement and reversal	Successfully demonstrates that weak measurement and its reversal can suppress amplitude-damping decoherence in a two-qubit system	Addresses the critical issue of maintaining entanglement, essential for quantum computing, cryptography, and teleportation.
9	Jeong et al. (2001) [27]	303	To investigate properties and applications of mixed entangled coherent channels for quantum teleportation and concentration	Shows entanglement of coherent states degrades in vacuum but does not vanish completely	Advances in the understanding of manipulating entangled coherent states for teleportation tasks.
10	Takahashi et al. (2010) [28]	276	To demonstrate entanglement distillation from Gaussian input states	Improves EPR correlations for two-photon subtracted states, beneficial for high-fidelity quantum teleportation	Provides a practical tool for extending quantum communication distances.

Table 8.2 Top 10 Most Productive Authors

Author	*Documents*	*Citations*	*Year*	*References*
Xiaojun Wen	6	57	2017.6667	[31, 29, 32–35]
Ying Guo	5	57	2016.6	[36–40]
Stefano Pirandola	4	2905	2016	[41–43, 19]
Fei Gao	4	184	2009.75	[44–46, 30]
Yan Chang	4	43	2019.25	[47–50]
Jian Li	4	22	2021	[51–54]
Yongzhi Chen	4	14	2021	[34, 33, 32, 31]
Junbin Fang	4	14	2021	[33, 34, 32, 31]
Qiao-Yan Wen	3	204	2009.25	[44, 55, 30]
Jonathan P. Dowling	3	181	2017.6667	[56–58]

Table 8.3 Countrywise Research Productivity

Sr.no.	*Country*	*Documents*	*Citations*	*Average Publication Year*	*Average Citations*
1	China	84	1404	2017.4048	16.7143
2	United States	34	3767	2016.6765	110.7941
3	India	27	321	2018.6667	11.8889
4	United Kingdom	25	4858	2013.5200	194.3200
5	Germany	14	4788	2013.8571	342.0000
6	Japan	14	786	2012.0714	56.1429
7	Brazil	8	53	2014.5000	6.6250
8	Canada	8	2898	2017.6250	362.2500
9	Italy	8	1160	2014.7500	145.0000
10	South Korea	8	971	2010.6250	121.3750

research productivity data are presented in Table 8.3. China maintains a dominant position globally with 84 documents, that is, 36.84% of the total documents. With 34 documents, the United States has maintained second position and massive influence with 3,767 citations, an enormous 110.79 citations per publication. After the United States, India has published 27 documents, which contribute to a citation per document of 11.89. The United Kingdom follows closely with 25 documents but has the highest total citations (4858), which demonstrates the value of its research and its global influence. Germany and Canada have published 14 and 8 documents with high average citations of 342 and 362.25, respectively. Japan has 14 documents and 786 citations; Italy and South Korea have 8 documents each, and respectable average citation counts of 145 and 121.38, respectively. Brazil and Canada, however, differ greatly, despite producing the same number of documents (8). Canada vastly outperforms Brazil in scholarly citations.

8.5.3.3 Most Productive Journals

Table 8.4 provides the distribution of publications for the most productive journals on QT in quantum cryptography over the years 2000 to 2024, while Figure 8.7 illustrates this data on line chart.



Figure 8.6 Productive countries.

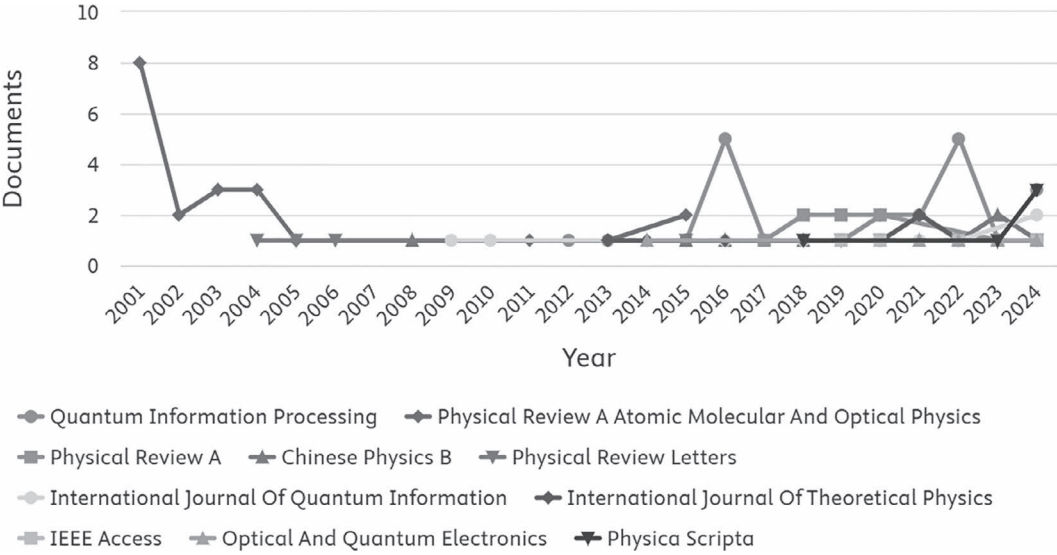


Figure 8.7 Productive journals.

Quantum Information Processing took the lead as the most active journal with a staggering count of 24 articles, which sharply increased starting in 2009 with 1 article, then 7 articles in 2013–2016, 5 articles in 2017–2020, and finally peaking at 11 articles from 2021 to 2024. *Physical Review A: Atomic Molecular and Optical Physics* significantly contributed in the earlier years with 16 articles between 2000 and 2004, followed by one article in 2005–2008, 1 from 2009 to 2012, and then 3 from 2013–2016, but no publications from 2016 to 2024. *Physical Review A* had no publications prior to 2013 but then became active, publishing seven papers in 2017–2020 and two in 2021–2024. *Physical Review Letters* had a total of nine papers: one in 2000–2004, two in 2005–2008, two in 2013–2016, and three in 2021–2024. *Chinese Physics B* had eight articles total, starting with 1 from 2005 to 2008 and

Table 8.4 Top Journals Publishing on QT in Quantum Cryptography

Journal Name	*2000–2004*	*2005–2008*	*2009–2012*	*2013–2016*	*2017–2020*	*2021–2024*	*Total*
Quantum Information Processing	0	0	1	7	5	11	24
Physical Review A: Atomic Molecular and Optical Physics	16	1	1	3	0	0	21
1.1.1. Physical Review A	0	0	0	0	7	2	9
1.1.2. Physical Review Letters	1	2	0	1	2	3	9
1.1.3. Chinese Physics B	0	1	0	2	1	4	8
International Journal of Quantum Information	0	0	2	1	1	4	8
International Journal of Theoretical Physics	0	0	0	3	2	3	8
1.1.4. IEEE Access	0	0	0	0	2	3	5
Optical and Quantum Electronics	0	0	0	1	0	4	5
1.1.5. Physica Scripta	0	0	0	0	1	4	5

increasing gradually to four in 2021 to 2024. *International Journal of Quantum Information and International Journal of Theoretical Physics* both added eight papers, the majority published between 2013 and 2024. Each of *IEEE Access*, *Optical and Quantum Electronics*, and *Physica Scripta* had five papers with no activity prior to 2013, indicating their more recent engagement with the topic.

8.5.4 What Are the Emerging Topics for QT within the Realm of Quantum Cryptography on the Basis of Keyword Analysis?

To answer this question, we have analyzed the keywords using the keywords overlay network. Figure 8.8 presents the overlay network of 320 keywords, in which each cluster is divided on the basis of average publication year. The yellow cluster in the network contains most of the keywords that have an average publication year of 2024. Table 8.5 presents the yellow cluster keywords and their total link strength in the keywords overlay network. Keywords such as quantum signature, quantum one-time pad, and quantum zero-knowledge mark the emergence of the new era of quantum-secured protocols. These systems use superposition and entanglement for absolute authentication, confidentiality, and integrity. In particular, the quantum one-time pad ensures perfect secrecy when employing truly random keys and states generated via teleportation. Additionally, quantum zero-knowledge schemes enable the implementation of proof systems while allowing sensitive data to remain hidden. Real-world quantum systems face significant challenges due to noise and imperfections. New research is focused on decoherence, channel infidelity, and diamond distance. These factors measure how accurately the state after transmission corresponds to the original and improve procedures for quantum state error correction, noise reduction, and recalibration.

The yellow cluster also highlights the increasing use of mathematical and cryptographic instruments for security enhancement. For example, quantum state discrimination, teleportation protocols, and secure resource allocation are fields where semi-definite programming is widely applied. Privacy amplification, especially in QKD and teleportation-based

Table 8.5 Yellow Cluster Keywords

Keywords	*TLS*	*APY*
3D (dd) Entangled State	3	2024
Anonymity	3	2024
Anonymous Communication	3	2024
Block Diagonal Matrices	5	2024
Changing the Destination	3	2024
Channel Infidelity	5	2024
Computational Complexity	6	2024
Decoherence	3	2024
Diamond Distance	5	2024
Distinctions and Similarities	2	2024
Empirical Study	3	2024
Entangled States	4	2024
Key Reuse	4	2024
Mach–Zehnder Interferometer	3	2024
Metaverse	4	2024
Optical Fiber Networks	6	2024
Privacy Protection	3	2024
Public-Key Cryptography	4	2024
Quantum Computing	6	2024
Quantum Detection	2	2024
Quantum Teleportation	6	2024
Quantum Proofs	6	2024
Quantum Signature	4	2024
Quantum One-Time Pad	4	2024
Quantum Physics	3	2024
Quantum Routing	3	2024
Quantum Simulation	4	2024
Quantum Steering	3	2024
Quantum Communication	3	2024
Quantum Zero-Knowledge	6	2024
Qubits	3	2024
Relativity	3	2024
Secure Interactions	4	2024
Semi-Definite Programming	5	2024
Quantum Secret	3	2024
Squeezed State	3	2024
Toeplitz Hash	4	2024
Trust Center	3	2024
χ State	4	2024

Abbreviations: TLS = total link strength, APY = average publication year.

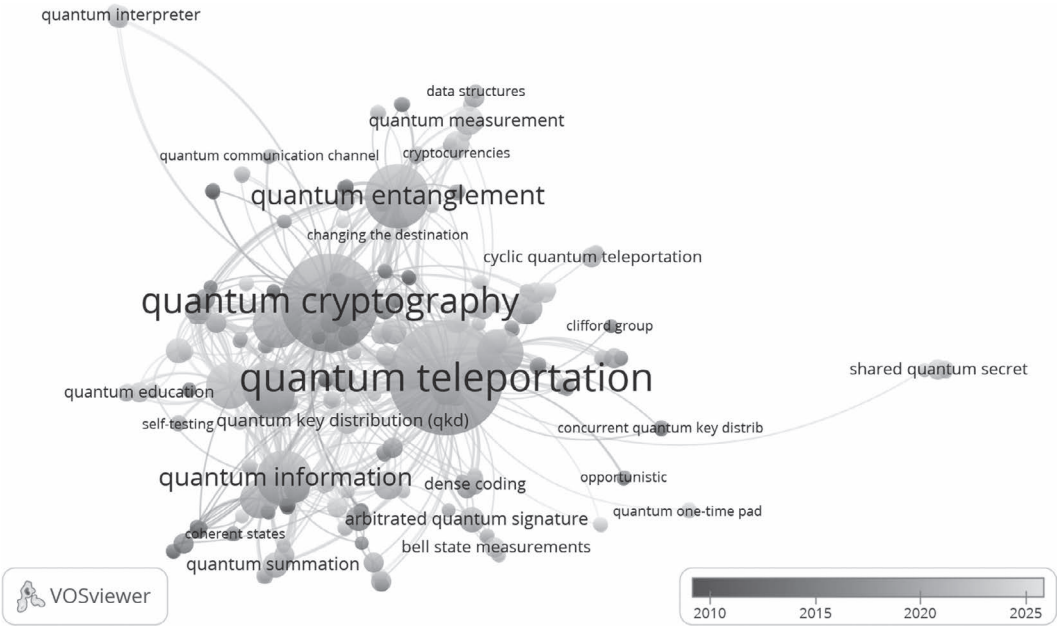


Figure 8.8 Keyword overlay network.

cryptographic systems, is another area where Toeplitz hash functions are frequently employed. The surge in digital communications brings into the spotlight privacy protection, anonymity, and anonymous communication. These areas often touch upon quantum zero-knowledge and secure interactions, indicating the emergence of a new class of cryptographic protocols focused on privacy, verification, and loss of security-sensitive infrastructure. At last, the keywords metaverse and relativity highlight a shift toward interdisciplinary research and fundamental study, while speculative areas of research include applying QT for secure interactions within the metaverse and the synchronization of the teleportation protocols in relativistic contexts.

8.6 CONCLUSION

This study provides the scientometric analysis of 228 articles retrieved from the Scopus database, spanning from 2000 to 2024. We have provided a detailed analysis by addressing fundamental research questions regarding the growth pattern of academic productivity, productive authors and nations, core journals, and emerging topics within the rapidly expanding interdisciplinary field. To effectively present our results, we employed tools such as VOSviewer and Microsoft Excel.

Analysis shows a sharp increase in research activity focused on QT in quantum cryptography, which tends to steeply rise starting in 2014 and peaks in 2024. This rise is in line with the general increase and booming interest in quantum communication systems, secure data transmission, and the development of infrastructure for quantum computing. This study illustrates the disciplinary focus of physics and astronomy and substantial inputs from engineering, computer science, mathematics, and material science, which shows the multidisciplinary applicability.

Citation network analysis revealed core documents and their authors, which strongly influenced the field's intellectual architecture. The most-cited pioneering documents were Horodecki et al. (2009) [18] and Weedbrook et al. (2012) [19], along with Pan et al. (2001) [20], underlining the key theoretical components of quantum entanglement, Gaussian states, and entanglement purification, which are important for QT.

China dominates the publication landscape as the most prolific country, while the United States, the United Kingdom, Germany, and Canada demonstrate higher impact through citation averages, showcasing the quality and influence of their contributions.

Quantum information processing has emerged as the most productive journal, suggesting an increasing interest toward application-driven research in the field, whereas *Physical Review A* and *Physical Review Letters* were the second- and third-most productive journals in the field, which provide essential knowledge for foundational research.

Moreover, emerging research themes have been identified through keyword overlay analysis, which includes topics like quantum signature, quantum one-time pad, quantum zero-knowledge, and quantum state discrimination. This reflects an emphasis on sophisticated quantum security protocols, as well as mechanism design for preserving privacy while interweaving quantum systems with future technologies like the metaverse and relativistic frameworks. Additionally, the recent developments in semi-definite programming and Toeplitz hashing fortify the bounds of mathematically defending quantum-secured communication, showcasing the growing application of advanced mathematics.

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Chapter 9

Quantum Phase Parameterized True Random Number Generation for Secure Communication

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9.1 INTRODUCTION

Randomness is a fundamental pillar of cryptographic security, underpinning critical operations such as key generation, encryption, authentication, and digital signatures. The integrity of these systems hinges on the quality of random numbers, where even subtle biases or patterns can render them vulnerable to exploitation. Randomness is the cornerstone of cryptographic security, enabling secure key generation, encryption, authentication, and digital signatures. The quality of random numbers directly impacts system resilience, as adversaries can exploit biases or patterns (Gururaja et al., 2025a). Classical pseudo-random number generators (PRNGs) rely on deterministic algorithms seeded with initial values, producing sequences that mimic randomness but are inherently predictable if the seed or algorithm is compromised (Goldreich, 2001). True random number generators (TRNGs), which exploit physical phenomena such as thermal noise, radioactive decay, or atmospheric fluctuations, offer greater unpredictability but are susceptible to environmental perturbations, potentially introducing correlations or biases (Vincent, 1970). Quantum random number generators (QRNGs) represent a paradigm shift, harnessing the inherent indeterminacy of quantum mechanics to deliver truly non-deterministic randomness with maximal entropy, making them ideal for high-security cryptographic applications (Bernhardt, 2019).

9.1.1 Background

The quest for reliable randomness has a rich history spanning centuries, evolving in tandem with advancements in computational and cryptographic technology. Early methods of random number generation relied on manual or mechanical processes, such as dice, coin flips, or lottery draws, which were inherently slow and prone to human bias. These techniques, dating back to ancient civilizations such as Mesopotamia and Egypt, were used for decision-making, divination, and games of chance but lacked the rigor required for modern cryptography. During the Renaissance, mathematicians such as Gerolamo Cardano developed early concepts of probability and compiled tables of random digits, although these methods remained labor-intensive and limited in scope. In the mid-20th century, the advent of digital computing spurred the development of algorithmic PRNGs, such as the linear congruential generator introduced by Lehmer in 1948, which produced sequences based on deterministic recurrence relations (Lehmer, 1948). These PRNGs, defined by the recurrence $x_{n+1} = (ax_n + c) \bmod m$, were efficient for early computational tasks, such as Monte Carlo simulations, but exhibited predictability due to their deterministic nature, with vulnerabilities including short periods (at most m) or statistical biases that could be exploited in cryptographic contexts. For example, if m is poorly chosen, the sequence may repeat after just a few iterations, allowing attackers to predict future outputs from partial samples.

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The introduction of physical TRNGs in the late 20th century marked a significant improvement, leveraging phenomena such as electronic noise, thermal fluctuations, or radioactive decay to generate randomness. Early TRNGs used devices like Geiger counters to measure the timing of radioactive decay events, producing bits based on the unpredictable intervals between decays. These systems, implemented in the 1960s and 1970s for scientific experiments, offered true randomness but required careful calibration to mitigate environmental influences, such as temperature drifts or electromagnetic interference, which could introduce subtle correlations. For instance, a TRNG relying on thermal noise might produce biased outputs if the ambient temperature fluctuates, leading to measurable deviations in bit distributions (e.g., a 48:52 ratio of 50:50). Moreover, their output rates, typically in the range of kilobits per second, were insufficient for modern high-speed cryptographic systems requiring megabits or gigabits per second.

The emergence of quantum mechanics in the early 20th century laid the theoretical foundation for QRNGs, with pioneering experiments in the 1980s and 1990s demonstrating the potential of quantum indeterminacy for randomness generation (Rarity et al., 1994). Quantum mechanics introduced the principle of inherent randomness, where the outcomes of measurements on quantum systems, such as the polarization of a photon or the spin of an electron, are fundamentally unpredictable. The 1920s witnessed the development of quantum theory by pioneers such as Heisenberg and Schrödinger, who established principles like superposition and measurement uncertainty. These concepts were formalized by the Heisenberg uncertainty principle, which states that certain pairs of properties (e.g., position and momentum) cannot be measured simultaneously with arbitrary precision. In the 1980s, experiments using single-photon detectors measured photon arrival times, generating random bits based on the probabilistic nature of photon emission. By the 2000s, advancements in quantum optics and photonics enabled the development of practical QRNGs capable of gigabit-per-second output rates, leveraging phenomena such as photon polarization or phase fluctuations (Huang et al., 2020). For example, a photon-based QRNG can measure the random polarization of single photons passing through a polarizing beam splitter, achieving rates of up to 68 Gbps, making it suitable for real-time cryptographic applications. These developments marked a turning point, positioning QRNGs as a viable solution for secure randomness in the digital age.

Classical PRNGs and TRNGs face significant limitations in the context of modern cryptography. PRNGs, such as the Mersenne Twister, produce sequences that are entirely deterministic once the seed is known, making them vulnerable to seed recovery attacks. For example, a poorly chosen seed or a compromised random number generator (RNG), as seen in the 2017 WannaCry ransomware attack, can lead to catastrophic security breaches (Schneier, 2018). In the WannaCry incident, attackers exploited predictable PRNG outputs to decrypt files, highlighting the risks of deterministic randomness in high-stakes applications. TRNGs, while less predictable, are susceptible to external factors such as temperature, electromagnetic interference, or ageing hardware, which can introduce subtle correlations. A notable case is the failure of a TRNG in a smart card system in the early 2000s, where environmental noise introduced biases that allowed attackers to predict encryption keys (Anderson, 2001). In this incident, temperature variations caused the TRNG to produce a skewed bit distribution, reducing the effective key space and enabling a brute-force attack. These vulnerabilities underscore the need for RNGs that are both unpredictable and resilient to external influences, a challenge that QRNGs address by leveraging quantum mechanics.

QRNGs leverage quantum properties, such as superposition, entanglement, and measurement uncertainty, to generate random outputs that are fundamentally unpredictable, even in principle (Nielsen & Chuang, 2010). Unlike classical TRNGs, QRNGs are less affected by external influences, ensuring consistent randomness quality across diverse operating

conditions. Superposition allows a quantum system to exist in multiple states, with measurement outcomes determined probabilistically. For instance, a qubit in a superposition state $1/\sqrt{2}$ ($|0\rangle + |1\rangle$) has a 50% probability of collapsing to $|0\rangle$ or $|1\rangle$ upon measurement, providing a source of true randomness. Entanglement enables the generation of correlated random bits, which is critical for applications such as quantum key distribution (QKD), where two parties can share perfectly correlated random sequences. Measurement uncertainty, governed by the Heisenberg uncertainty principle, ensures that quantum states cannot be predicted or cloned, enhancing security against adversarial attacks. A 2-qubit entangled state such as $1/\sqrt{2}$ ($|00\rangle+|11\rangle$) yields correlated outcomes upon measurement, with each qubit producing a random bit that matches its partner's result, a property exploited in protocols like BB84.

The mathematical framework of QRNGs is rooted in quantum mechanics. A single qubit in a superposition state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|\alpha|\ 2 + |\beta|\ 2 = 1$, yields probabilities $|\alpha|2$ and $|\beta|2$ for outcomes $|0\rangle$ and $|1\rangle$, respectively. For an equal superposition, $\alpha = \beta = 1/\sqrt{\ }2$, each outcome has a 50% probability, maximizing entropy at 1 bit per measurement. An n-qubit system in a fully superposed state, such as $\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle$, produces n bits per measurement, with $2n$ possible outcomes, scaling entropy exponentially. Phase modulation, as used in advanced QRNGs, introduces complex amplitudes (e.g., $1/\sqrt{\ }2$ ($|0\rangle + i|1\rangle$) after applying an S gate, enhancing interference patterns and potentially increasing resistance to classical simulation. In contrast, PRNGs, such as the linear congruential generator, are predictable due to their finite period, while TRNGs may exhibit correlations measurable through statistical tests, like autocorrelation or chi-squared tests. They often fail tests like NIST SP 800-22 when environmental conditions fluctuate (Rukhin et al., 2001).

Research classifies QRNGs into three main categories: practical, self-testing, and device independence, based on the level of trust required in the underlying hardware (Ma et al., 2016). Practical QRNGs, built on trusted devices, achieve high generation rates by measuring quantum processes like photon polarization, phase fluctuations, or qubit collapses (Li et al., 2024). These systems prioritize performance, often exceeding gigabits per second (e.g., 68 Gbps for photonic QRNGs), and are suitable for commercial applications, such as secure key generation in financial systems. Self-testing QRNGs incorporate protocols to verify randomness without requiring full device characterization, thereby enhancing security in partially trusted environments (Abbott et al., 2014). Authors employed techniques such as randomness expansion, where a small amount of initial randomness is amplified into a larger, certified random sequence, ensuring reliability even if the hardware is not fully trusted, achieving rates of 1–10 Mbps. Device-independent QRNGs, certified by Bell's theorem, rely on quantum non-locality to guarantee randomness, offering the highest theoretical security (Abbott et al., 2012). However, their complex experimental setups, which require entangled particle pairs and violate Bell inequalities, limit their scalability, with current systems achieving rates below 1 Mbps, thereby confining them to research settings.

The chapter centers on the development, implementation, and validation of a novel QRNG, the quantum phase-parameterized true random number generator (QPPTRNG), using IBM's Qiskit platform. It focuses on leveraging Qiskit's modular architecture, Qiskit Terra for circuit design, Qiskit Aer for simulations, Qiskit IBMQ Provider for hardware access, and Qiskit Experiments for noise analysis and executing QRNG circuits. The chapter examines the design of quantum circuits that utilize phase modulation and superposition to produce high-entropy random bits, with a focus on scalability and efficiency for practical deployment. It includes rigorous statistical validation of the QPPTRNG's output using industry-standard test suites, such as NIST SP 800-22, NIST SP 800-90B, Dieharder, and ENT, ensuring its suitability for cryptographic applications (Walker, n.d.). The scope extends to applications

in post-quantum cryptography, secure communication (e.g., QKD protocols such as BB84), and quantum-enhanced stochastic modelling, including Monte Carlo simulations in quantum chemistry and finance.

Additionally, the chapter examines the role of Qiskit in democratizing QRNG research, fostering collaboration through open-source contributions, and supporting educational initiatives in quantum computing. It also addresses challenges in QRNG deployment, such as hardware noise and device trust. It proposes future directions for fault-tolerant and hybrid quantum-classical systems, aligning with the broader goal of advancing quantum-safe security solutions in the quantum era.

9.1.2 Problem Definition

The problem addressed in this chapter is the inadequacy of classical randomness generation methods: PRNGs and TRNGs meet the security and performance demands of modern cryptographic systems, particularly in the context of quantum computing threats. PRNGs, despite their computational efficiency, are deterministic and predictable if the seed or algorithm is compromised, as demonstrated by vulnerabilities in systems like the Mersenne Twister, where seed recovery attacks can reconstruct the entire sequence (Schneier, 2018). For example, a PRNG with a 32-bit seed has only 232 possible sequences, a space that can be exhaustively searched by modern computers, rendering it insecure for cryptography. TRNGs, while offering true randomness, are susceptible to environmental biases, such as temperature or electromagnetic interference, which can skew bit distributions and introduce correlations detectable through statistical tests (Anderson, 2001). These correlations, often with deviations of 1–5% from uniformity, compromise the security of cryptographic keys, as seen in the smart card failures of the early 2000s.

The rise of quantum computing exacerbates these issues, as classical cryptographic algorithms like RSA and elliptic curve cryptography are vulnerable to quantum attacks. Shor's algorithm, which efficiently factors large integers and solves discrete logarithm problems on a quantum computer, poses a threat to the security of these systems (Williams, 2011). A 2048-bit RSA key, secure against classical attacks, could be factored in hours on a sufficiently large quantum computer, necessitating the development of quantum-resistant algorithms. Grover's algorithm, which provides a quadratic speedup for unstructured search problems, reduces the effective key strength of symmetric ciphers, requiring longer keys (Watrous, 2018). For instance, a 128-bit AES key offers only 64-bit security against Grover's algorithm, doubling the required key length to maintain security. These quantum threats highlight the critical need for high-quality randomness in post-quantum cryptography, where secure key generation and encryption rely on unpredictable random bits.

Moreover, modern applications require high-speed random number generation, often in the range of gigabits per second, to support real-time systems such as secure communication networks, financial transactions, and cloud computing. Classical TRNGs struggle to meet these rates, typically producing kilobits per second, while PRNGs, though faster, lack the necessary unpredictability. QRNGs offer a solution by leveraging the principles of quantum mechanics. Still, their practical deployment faces challenges, including hardware noise (e.g., 0.1–1% gate errors), limited qubit coherence times (50–200 ffs), and the need for trusted devices. The QPPTRNG aims to address these issues by providing a scalable, efficient, and cryptographically secure QRNG design implemented on IBM's Qiskit platform, meeting the demands of the quantum era.

The rise of quantum computing poses both challenges and opportunities for cryptographic systems. Classical cryptographic algorithms, such as RSA and elliptic curve cryptography, rely on computational hardness assumptions that are vulnerable to quantum attacks. Shor's

algorithm poses a threat to the security of these systems. In contrast, Grover's algorithm reduces the effective key strength of symmetric ciphers, necessitating longer keys or the use of quantum-resistant algorithms. These quantum threats underscore the crucial role of high-quality randomness in post-quantum cryptography, where QRNGs offer a robust foundation for secure key generation and encryption. QKD protocols, such as BB84, rely on QRNGs to generate random bits for key exchange, thereby ensuring security against eavesdropping through the principles of quantum mechanics (DiVincenzo, 2000). In BB84, QRNGs generate random polarization states (e.g., horizontal, vertical, diagonal, and anti-diagonal) to encode bits, with quantum no-cloning ensuring that any eavesdropping attempt introduces detectable errors, thereby achieving information-theoretic security.

Beyond cryptography, QRNGs have transformative applications in diverse fields. In quantum chemistry, high-quality randomness enhances the accuracy of Monte Carlo simulations for molecular modelling, enabling precise predictions of chemical reactions (Pathak et al., 2025). For instance, QRNGs can improve the sampling efficiency of variational quantum eigen solvers reducing errors in calculating molecular ground states, with applications in drug discovery. In optimization, algorithms such as simulated annealing rely on random sampling, where unbiased randomness from QRNGs can enhance convergence to global optima. In machine learning, QRNGs enhance randomized neural network initialization and stochastic gradient descent, thereby reducing bias and improving model generalization (Goodfellow et al., 2016). Stochastic simulations in finance, physics, and biology, such as option pricing or protein folding, also leverage QRNGs to ensure reliable statistical outcomes. In financial modelling, QRNGs improve Monte Carlo methods for pricing complex derivatives, whereas the classical RNGs may introduce systematic errors due to periodicity (Unruh, 2021).

QRNGs also play a critical role in emerging technologies. In blockchain, QRNGs ensure the unpredictability of nonces and consensus mechanisms, thereby enhancing security against attacks such as double-spending, where predictable randomness could allow malicious actors to manipulate transactions. Secure multi-party computation, used in privacy-preserving data analysis, relies on random inputs to guarantee fairness and confidentiality, with QRNGs providing a trusted source of randomness. In quantum machine learning, QRNGs enhance quantum neural networks by providing unbiased initial weights, reducing training time, and improving accuracy, with potential applications in natural language processing and image recognition. In gaming and simulations, QRNGs ensure fairness and unpredictability, which are critical for applications such as online gambling and virtual reality environments. In the emerging field of quantum internet, QRNGs support secure communication protocols by generating random bits for entanglement-based QKD, enabling global quantum networks (DiVincenzo, 2000). In digital forensics, QRNGs enhance the integrity of random sampling for evidence analysis, ensuring unbiased results in legal investigations. These diverse applications highlight the potential of QRNGs to revolutionize technology across various sectors, from security to scientific discovery.

The development of QRNGs has been accelerated by the emergence of quantum computing platforms, which provide accessible tools for designing, simulating, and executing quantum circuits. IBM's Qiskit, an open-source Quantum Information Science Kit launched in 2017, is a leading framework that empowers users to explore quantum algorithms, including QRNGs, on cloud-based quantum hardware (Aleksandrowicz et al., 2019). With over 6 million installations and 3 trillion circuits executed by June 9, 2025, Qiskit's modular architecture, comprising Qiskit Terra for circuit design, Qiskit Aer for simulations, Qiskit IBMQ Provider for hardware access, and Qiskit Experiments for noise analysis, supports a wide range of applications, from educational initiatives to cutting-edge research (Abraham et al., 2019; IBM Quantum Experience, n.d.). Qiskit Terra enables users to construct quantum circuits using gates such as Hadamard or rotation gates (e.g., RY, S). At the same time,

Qiskit Aer provides high-performance simulators for validating circuit behavior under ideal or noisy conditions, with simulation times for a 10-qubit circuit ranging from 0.1 to 1 second on a standard CPU. The IBMQ Provider connects users to IBM's quantum processors, ranging from 5-qubit systems for public access to 127-qubit systems for research, with calibration data ensuring transparency (e.g., T1/T2 times of 50–200 ffs, gate errors of 0.1–1%). Qiskit experiments facilitate noise characterization and error mitigation, which are critical for reliable QRNG performance on noisy hardware, achieving fidelity improvements of 20–50% through techniques such as zero-noise extrapolation.

Qiskit's accessibility has democratized quantum computing, fostering a global community through resources such as the Qiskit Textbook, certification programs, and summer schools. Universities have adopted it as a standard tool for teaching quantum mechanics, information theory, and algorithm design (Jones et al., 2020). The Qiskit Textbook provides tutorials on quantum algorithms, including QRNGs, while certification programs validate expertise, with over 10,000 certified users expected by 2025. Summer schools and hackathons engage thousands of participants annually, driving innovation in quantum apply QRNGs (action), such as QRNGs, for secure voting systems. In QRNG research, Qiskit facilitates the rapid prototyping of circuits, such as those utilizing Hadamard gates for superposition or rotation gates for phase modulation, which are validated by statistical test suites like NIST and Dieharder (Marsaglia, 1995; NIST, n.d.). Qiskit's ecosystem continues to evolve with advancements, such as Qiskit Runtime, which enhances performance through optimized circuit execution and error mitigation (Banerjee, 2026). Qiskit runtime reduces latency by up to 100 times, enabling near-real-time QRNG deployment. Meanwhile, error mitigation techniques improve output fidelity on noisy intermediate-scale quantum hardware, reducing error rates by 20–50%.

Challenges in QRNG deployment include hardware noise, with IBM's qubits exhibiting coherence times of 50–200 femtoseconds and gate errors of 0.1–1%, which introduce biases in bit distributions. Readout errors (5%) require mitigation, such as matrix inversion, to ensure accuracy, correcting deviations to within 0.1% of ideal probabilities. Scalability to large qubit counts increases transpiration complexity, as limited qubit connectivity necessitates the use of SWAP gates, which adds 1–2% error per gate. Practical QRNGs rely on trusted hardware, raising concerns about device integrity during self-testing, while device-independent QRNGs sacrifice speed for security. Post-processing, such as randomness extraction, adds computational overhead but ensures cryptographic quality, achieving min-entropy scores above 0.99. Future advancements aim to address these challenges, with fault-tolerant QRNGs utilizing error correction codes, such as surface codes, to mitigate noise and hybrid quantum-classical systems integrating QRNGs with classical post-processing for enhanced applications. For example, a hybrid system could combine QPPTRNG outputs with classical entropy extraction to meet stringent cryptographic standards, achieving rates of 50–100 Mbps with near-perfect randomness.

This chapter presents the QPPTRNG, a novel QRNG implemented on IBM's Qiskit platform, designed to address the growing demand for secure randomness in the quantum era. The QPPTRNG employs a scalable circuit architecture, applying RY ($\pi/2$) and S gates to multiple qubits to create quantum states with maximal unpredictability, which is enhanced by phase modulation (Gururaja et al., 2025b). The RY ($\pi/2$) gate generates an equal superposition of computational basis states, ensuring uniform measurement probabilities. In contrast, the S gate introduces a $\pi/2$ phase shift, enriching the state's complexity through interference patterns. This phase-parameterized approach distinguishes the QPPTRNG from simpler Hadamard-based QRNGs, offering higher entropy and resistance to classical predictability (Nielsen, 2002). The circuit's scalability allows for exponential growth

in output size with additional qubits, while its efficiency, requiring only two gates per qubit, supports practical deployment on IBM's superconducting quantum processors. The QPPTRNG's randomness has been thoroughly validated using industry-standard statistical test suites, including NIST SP 800-22, NIST SP 800-90B, Dieharder, and ENT, confirming its suitability for cryptographic applications (Walker, n.d.). These tests demonstrate uniform bit distributions (e.g., 50:50 within 0.1% deviation), negligible correlations (e.g., autocorrelation below 0.01), and near-maximal entropy (e.g., 0.9983–0.9985), underscoring the bitstream's statistical robustness. Compared to classical TRNGs and PRNGs, such as those based on cellular automata, the QPPTRNG excels by leveraging quantum measurement collapse, rendering its outputs inherently non-deterministic and immune to cryptanalysis (Oskin et al., 2002). The QPPTRNG's phase-based approach aligns with advanced QRNG designs that exploit phase noise or fluctuations, achieving high generation rates and resilience against classical noise. The QPPTRNG's scalability allows for exponential growth in output size by adding qubits. At the same time, its potential to incorporate entangling operations, such as CNOT or CZ gates, enables the generation of correlated randomness for specialized applications, including QKD or quantum-resistant encryption. Its implementation on Qiskit ensures accessibility for researchers and practitioners, fostering collaborative development through open-source contributions. The QPPTRNG builds on a rich foundation of QRNG research, addressing key challenges such as generation speed, device trust, and integration with quantum hardware. Its applications extend to post-quantum cryptography, secure communication, and quantum-enhanced stochastic modelling, where high-entropy randomness is critical. Future directions in QRNG research aim to enhance generation rates, develop fault-tolerant designs, and integrate QRNGs into scalable quantum-classical systems. Fault-tolerant QRNGs, leveraging quantum error correction codes such as surface codes, can mitigate noise on large-scale quantum hardware, ensuring reliable randomness even in the presence of decoherence. Integration with quantum-classical systems, such as hybrid quantum-classical algorithms for machine learning, could expand QRNG applications to new domains, including quantum internet and digital forensics. Qiskit's ongoing advancements, including Qiskit Runtime for optimized execution and error mitigation techniques, position it as a key enabler for these goals. QPPTRNG's open-source nature aligns with the principles of open science, promoting transparency, reproducibility, and collaborative progress in quantum computing. By addressing the critical need for secure randomness in the quantum era, the QPPTRNG establishes a robust framework for cryptographic protocols, secure key generation, and quantum-safe communication systems, paving the way for practical quantum-based randomness sources in an increasingly quantum-driven world.

Objectives of the Chapter

The objectives of the chapter are as follows:

- To present the QPPTRNG, a novel QRNG design implemented on IBM's Qiskit platform, leveraging phase modulation for enhanced randomness;
- To validate the QPPTRNG's performance through rigorous statistical testing using standard suites like NIST SP 800-22 and Dieharder, demonstrating its suitability for cryptographic applications;
- To explore the QPPTRNG's applications in post-quantum cryptography, secure communication (e.g., QKD), and quantum-enhanced simulations, highlighting its versatility;

- And, to foster collaborative development by providing an open-source implementation on Qiskit, encouraging researchers and practitioners to refine and extend the QPP-TRNG for diverse use cases.

Organization of the Chapter

The rest of the chapter is organized as: Section 9.2 discusses the proposed quantum phase-parameterized gate mathematical model. Section 9.3 enlightens results and discussion. Section 9.4 focuses on comparative analysis and discussion. And, finally, section 9.5 concludes the chapter with future scope.

9.2 PROPOSED QUANTUM PHASE-PARAMETERIZED GATE MATHEMATICAL MODEL

The custom quantum gate introduces a phase shift parameterized by a continuously tunable variable θ, exploiting inherent quantum randomness to generate true random numbers. This phase shift is applied to a qubit initialized in a superposition, resulting in a probabilistic measurement outcome that is influenced by θ, yet fundamentally unpredictable due to the principles of quantum mechanics. In this section, the mathematical model of the QPPTRNG architecture is highlighted. The generation of quantum true randomness relies on quantum superposition and the probabilistic nature of measurement outcomes in a quantum system. The proposed model utilizes the $RY(\pi/2)$ gate and S gate in a compositional architecture to construct states with unpredictability. A formal analysis of the gates, their combined operations, and how their interaction leads to a non-deterministic entropy state suitable for quantum random number generation is presented.

9.2.1 RY(π/2) Gate Properties

The RY gate is a parameterized and single-qubit rotation gate around the Y-axis of the Bloch sphere. It performs a rotation of a qubit's state vector around the Bloch sphere by a fixed $\pi/2$ radian. This operation transforms a basis state into an equal superposition of both computational basis states. From a physical state, this superposition introduces intrinsic uncertainty into the system. When measured, a qubit in this state cannot deterministically yield a known outcome; it probabilistically collapses to either 0 or 1. This feature is fundamental to quantum randomness as the measurement outcome is not a result of hidden variables but inherent indeterminacy embedded in quantum mechanics (Nielsen & Chuang, 2010). The RY($\pi/2$) gate prepares the quantum system in a maximally uncertain state. This lays the foundation for randomness by ensuring that the qubit does not produce one outcome over another; instead, the final result is decided at the moment of measurement, governed by quantum probability. The matrix representation of the RY gate:

$$RY(\theta) = \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix} \tag{9.1}$$

For $\theta = \frac{\pi}{2}$, the RY gate becomes:

$$RY\left(\frac{\pi}{2}\right) = \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{9.2}$$

This transformation rotates the quantum state from the computational basis state $|0\rangle$ to a superposition state.

$$RY\left(\frac{\pi}{2}\right)|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = |+\rangle \tag{9.3}$$

$$RY\left(\frac{\pi}{2}\right)|1\rangle = \frac{1}{\sqrt{2}}\left(-|0\rangle + |1\rangle\right) \tag{9.4}$$

These superpositions are the key to randomness, as measurement outcomes in the computational basis for these states are probabilistic, with equal or phase-parameterized probabilities depending on the gate combinations. This transformation is crucial in quantum algorithms that require controlled state preparation and is widely used in parameterized quantum circuits.

9.2.2 S Gate and Phase Shift Operation

The S gate is a Clifford gate and is also known as a phase gate. It applies a $\frac{\pi}{2}$ phase shift to the quantum state. This quantum gate introduces a fixed, controlled phase shift to one component of a quantum state. While this gate does not directly affect the probabilities observed upon measurement in the computational basis. The phase introduced by the S gate modifies the internal structure of the quantum state in the complex domain. This change becomes significant in multi-qubit systems or when measurements are performed on different bases. The phase shift influences interference and entanglement properties, adding another layer of complexity that enhances the unpredictability of outcomes. This enriched quantum state, characterized by both amplitude and phase complexity, yields output randomness that is irreducible by classical statistical models, making it ideal for cryptographic-grade true random number generation.

In the QPPTRNG, the S gate is applied after the RY(π/2) gate to each qubit. This sequence results in quantum states that not only exhibit amplitude superposition but also include nontrivial phase relationships. These phase-enriched superpositions are more robust in resisting classical predictability and produce randomness that is not just probabilistic but also structurally complex at the quantum level. It is represented as

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \tag{9.5}$$

It leaves $|0\rangle$ unchanged and maps $|1\rangle$ to $i|1\rangle$, introducing a relative phase shift between the basis states. This phase does not directly affect the measurement in the computational basis but rather alters interference patterns in superpositions when combined with other gates.

Applying the S gate after the RY gate alters the relative phase of the amplitudes:

$$S \cdot RY\left(\frac{\pi}{2}\right)|0\rangle = S\left(\frac{1}{\sqrt{2}}|0\rangle + |1\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) \tag{9.6}$$

This quantum state maintains an equal probability upon measurement, forming a complex phase of information that impacts multi-qubit interference and correlation.

9.2.3 Combined RY-S Gate Operation and State Vector Analysis

The combination of RY(π/2) followed by the S gate produces a quantum state that influences both amplitude-balanced and phase-shifted. This two-step transformation is central to QPPTRNG architecture. This composite gate sequence maps each qubit into a quantum state residing on the equator of the Bloch sphere, where measurement outcomes are maximally uncertain. States on the Bloch sphere equator have equal projection onto the |0⟩ and |1⟩ basis states, ensuring equal measurement probability. However, differing phase relationships embedded within these equatorial states result in different quantum interference effects when extended to multi-qubit or entangled scenarios. The phase shift introduced by the S gate ensures that, even though the measurement probabilities produce symmetrical results, the underlying quantum state retains unique internal features. These hidden features are not detectable from a single measurement and reduce potential predictability when sequences of measurements or correlated states are considered. The significance of this design creates a mechanism where, even if the input to the system is known, the outcome of measurements cannot be reproduced or reverse-engineered. It fundamentally leverages the unpredictability of quantum mechanics and the no-cloning theorem to produce results that are theoretically irreproducible by any classical or adversarial attack.

The full composite operation $U = S \cdot RY\frac{\pi}{2}$. The action on the computational basis states yields:

$$U|0\rangle = S \cdot RY\left(\frac{\pi}{2}\right)|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right);\ U|1\rangle = S \cdot RY\left(\frac{\pi}{2}\right)|1\rangle = \frac{1}{\sqrt{2}}\left(-|0\rangle + i|1\rangle\right) \tag{9.7}$$

These quantum gates form a resulting vector that is a complex-valued superposition with phase-parameterized amplitude contributions. Quantum states that are neither real-valued nor symmetric in amplitude or phase. Measurement of such states in the computational basis produces equiprobable outcomes with uncorrelated, high-entropy phase-modulated randomness. The resulting measurement outcomes represent quantum-originated bitstreams with high min-entropy, suitable for cryptographic randomness extraction without requiring extensive post-processing. Applying this operation to each qubit in an n-qubit register, when scaled to multi-qubit systems, creates an overall state that becomes an entangled and separable superposition depending on the circuit configuration. This results in a Hilbert space of 2^n possible outcomes, and each is probabilistically determined upon measurement.

9.2.4 Pre-Measurement Analysis of QPPTRNG

Figure 9.1 illustrates the core quantum circuit architecture employed in the proposed QPPTRNG, designed to scale efficiently across n qubits. Each qubit, denoted as $q0$ through $qn - 1$, is initialized in the ground state |0⟩ and subjected to a uniform sequence of quantum gates that together produce a high-entropy quantum state.

The first operation applied to each qubit is the RY(π/2) gate, a rotation about the *Y*-axis of the Bloch sphere by π/2 radians. This operation transforms the deterministic basis state into a balanced superposition state:

$$|\psi\rangle = RY\left(\frac{\pi}{2}\right)|\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \tag{9.8}$$

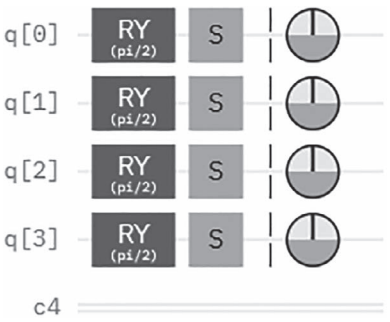


Figure 9.1 Quantum circuit diagram of the QPPTRNG architecture with four qubits, showing the application of RY(π/2) and S gates followed by *Z*-basis measurements.

This superposition ensures that the qubit, when measured, has a 50% probability of collapsing to either $|0\rangle$ or $|1\rangle$, which is a fundamental property leveraged for generating quantum randomness. Immediately following the RY gate, each qubit is passed through an S gate, also known as the phase gate, which introduces a relative phase shift of $\pi/2$ to the $|1\rangle$component of the qubit's state. Though this phase does not affect the individual measurement probabilities in a standard basis, it enriches the underlying quantum state space. It introduces more complex probabilistic interference in multi-qubit systems, which is especially beneficial for enhancing entropy when extended to entangled or correlated states.

After the RY and S gate operations, all qubits are measured in the computational (*Z*) basis. These measurements collapse the quantum states into classical bits—either 0 or 1—yielding *n* independent, non-deterministic output bits per circuit run. This structure is highly parallel, with each qubit operating independently yet identically, enabling the architecture to be both modular and scalable. Although the gates operate independently on each qubit, the resulting quantum state exhibits global phase-dependent complexity. This complexity is not due to entanglement in the traditional sense but rather to phase correlations that affect the interference structure of the joint state vector. Because each measurement result stems from an irreducibly random quantum process (rather than algorithmic pseudo-random logic), the resulting bitstrings are intrinsically non-reproducible and unpredictable. Moreover, by varying the number of qubits (*n*), this design allows for flexible throughout in random bit generation, as the system scales to n qubits, the dimensionality of the state space grows exponentially (2^n basis states), amplifying the complexity of the interference landscape and increasing the difficulty of reverse-engineering or simulating the output classically, which is a critical advantage in cryptographic systems demanding high entropy and high-speed generation.

This circuit effectively captures the essence of quantum mechanical indeterminacy through the RY(π/2) and S gate combination, offering a minimal yet powerful configuration for generating statistically robust and secure random numbers. Its simplicity, hardware efficiency, and scalability make it ideal for integration into practical QTRNG implementations, including those running on real quantum devices such as IBM quantum systems.

Figure 9.2 visualizes the complex quantum state distribution resulting from the execution of the QPPTRNG circuit on a 4-qubit register. It represents the output state vector on a multi-dimensional Bloch sphere (or complex sphere projection), where each ray from the origin corresponds to one of the $2n = 16$ computational states. The direction and length of each vector illustrate the amplitude (magnitude and phase) of the corresponding basis state's contribution to the overall quantum state, while the color ring (bottom-left) encodes the

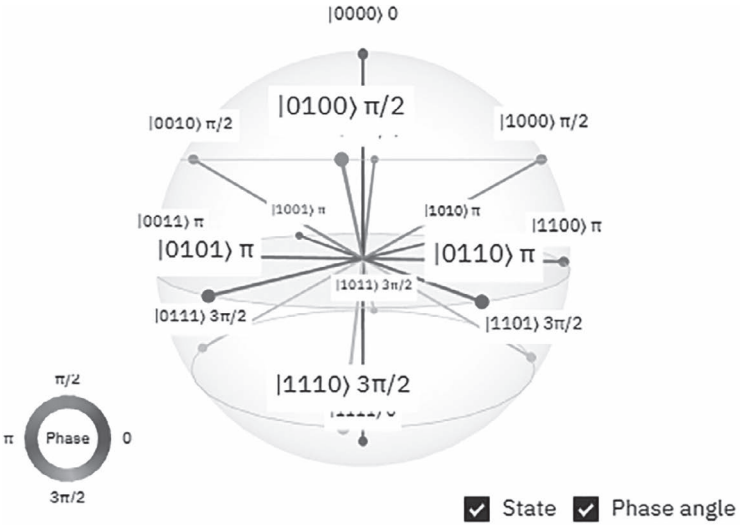


Figure 9.2 Bloch sphere representation of a multi-qubit quantum state after gate operations, illustrating phase angles assigned to each computational basis state.

relative phase angle from 0 to 2π, segmented into quadrants (0, $\pi/2$, π, $3\pi/2$). This color-coded phase information is crucial in illustrating the impact of the S gates applied after the RY($\pi/2$) rotations in the circuit, which introduces controlled $\pi/2$ phase shifts to the $|1\rangle$ component of each qubit.

For instance, basis states like $|0000\rangle$ appear with phase angle 0 (blue), while states such as $|0100\rangle$ $|1000\rangle$ and $|0010\rangle$ exhibit a $\pi/2$ phase (magenta), and others like $|0101\rangle$, $|0110\rangle$, or $|1001\rangle$ carry a π phase (red). States such as $|0111\rangle$ and $|1110\rangle$ reflect phase angles approaching $3\pi/2$ (yellow-brown), showing deeper phase progression due to multiple S gates acting simultaneously across qubits set to $|1\rangle$. These diverse phases collectively increase the internal complexity of the quantum state, thereby enhancing the entropy and uniformity of measurement probabilities when observed on a computational basis.

This multi-phase superposition ensures a non-trivial quantum interference landscape, where destructive and constructive interferences, governed by phase relationships, further randomize the measurement outcomes. The absence of dominance by any single basis state, evidenced by the relatively balanced vector length, demonstrates that no specific outcome is favored. It reinforces the true randomness property of the generator. This serves as a powerful validation of the quantum unpredictability and entropy amplification introduced by combining RY($\pi/2$) and S gates, ultimately ensuring that the QPPTRNG's output bitstrings maintain uniform probability, zero bias, and high entropy—key attributes confirmed in statistical test results, such as those from NIST, Dieharder, and ENT.

Figure 9.3 represents the amplitude distribution of the 16 computational basis states $|0000\rangle$ to $|1111\rangle$ generated from a 4-qubit quantum circuit composed of sequential Ry $\left(\frac{\pi}{2}\right)$ and S gates applied to each qubit. The horizontal axis denotes the basis states, while the vertical axis shows the corresponding magnitude (or absolute value) of the amplitude of each state in the final superposition. As expected from a well-formed QRNG, the amplitudes are uniform across all 16 states, each having a value of $\frac{1}{4}$, which, when squared, gives a probability of $\frac{1}{16} \approx 0.0625$. This perfectly flat distribution verifies that the output state is maximally mixed in the computational basis, with no bias toward any individual state.

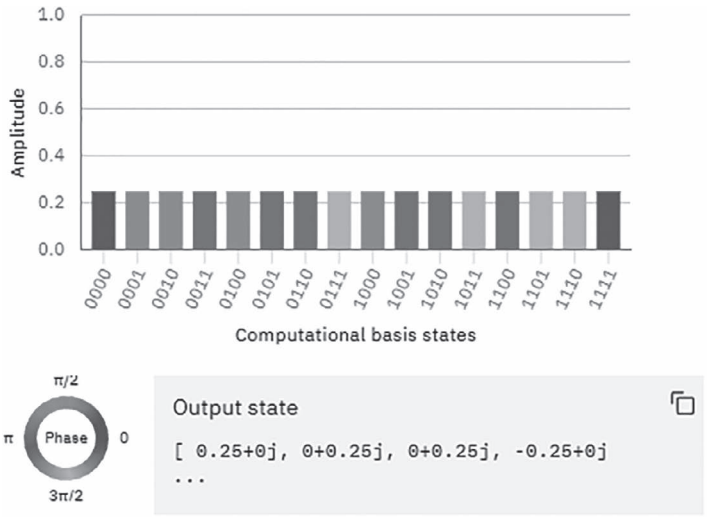


Figure 9.3 Amplitude distribution of computational basis states in the QPPTRNG output, showing near-uniform superposition.

Moreover, the color coding of each bar aligns with the quantum phase of the corresponding state derived from the phase wheel (bottom-left). Each color represents a different phase angle in the complex plane (0, $\pi/2$, π, $3\pi/2$), imparted by the cumulative effects of the S gates, which rotate the phase of the $|1\rangle$ component of each qubit by $\pi/2$ radians. For example, some states, such as $|0000\rangle$, have no phase (blue), while others like $|0001\rangle$, $|0010\rangle$, and $|0100\rangle$ possess a $\pi/2$ phase (magenta), and states such as $|0101\rangle$ or $|1001\rangle$ reach π or $3\pi/2$ (red or gold), respectively. These imaginary and complex values in the output state vector (e.g., $+0.25j$, $-0.25 + 0j$) confirm that the system is in a coherent superposition with diverse relative phases rather than a real-valued distribution, adding an essential dimension to entropy and non-determinism.

This graphical representation strongly supports the claim that the QPPTRNG achieves uniform probability with phase randomness, contributing to both amplitude and phase entropy. The uniform amplitude across all computational states validates the ideal superposition property. At the same time, the diverse phase spectrum ensures that the quantum interference pattern during measurement does not introduce correlation, predictability, or bias, which are crucial for cryptographic and randomness quality standards. Collectively, this figure confirms the theoretical and empirical soundness of the QPPTRNG design, reinforcing earlier statistical test results (NIST, Dieharder, ENT) that showed consistently high entropy and unpredictable bitstream generation.

Figure 9.4 presents a probability distribution histogram for the output of a 4-qubit quantum circuit after measurement, highlighting how each computational basis state is sampled from the final quantum state. The x-axis represents all 16 basis states from $|0000\rangle$ to $|1111\rangle$, while the y-axis indicates the observed probability (in percentage) of each state occurring upon measurement. As shown, each basis state exhibits a nearly identical probability of approximately 6.25% $\frac{1}{16}$, which aligns exactly with the expected outcome of an ideal uniform superposition state. This distribution further confirms that the circuit has successfully generated a perfectly balanced quantum state, where each classical outcome is equally likely. Such a result demonstrates no statistical bias or skewness, which is critical for the performance of QPPTRNG.

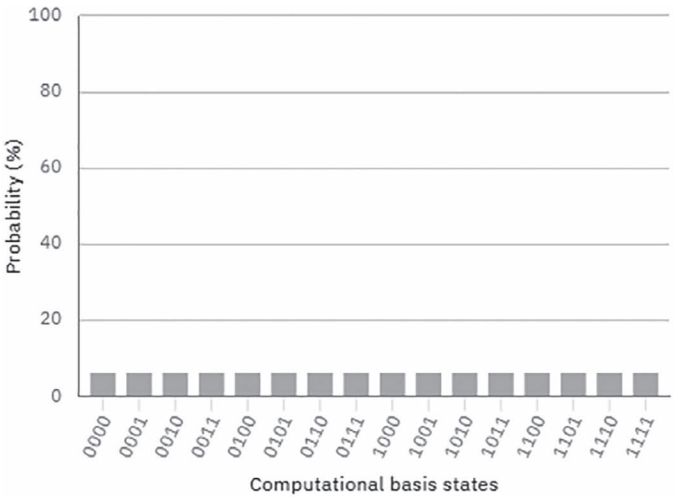


Figure 9.4 Probability distribution of measurement outcomes across all 4-qubit basis states, demonstrating balanced randomness.

The consistency of this result with earlier figures, such as the amplitude distribution and Bloch sphere phase encoding, reinforces the robustness of the circuit's design. The uniformity in measurement probabilities reflects the absence of decoherence, phase drift, or gate-induced errors, implying high quantum gate fidelity and minimal noise during circuit execution. Furthermore, this behavior aligns well with the high entropy scores and randomness tests (e.g., ENT, NIST 800-22) previously analyzed, validating that the QPPTRNG architecture reliably produces unbiased, high-entropy random outputs with both amplitude and phase randomness preserved. This probability plot serves as a strong empirical confirmation of the uniformity and unpredictability of the quantum-generated output, underscoring the effectiveness of the QPPTRNG in real-world applications of randomness generation and quantum cryptography.

Before any measurement is performed, the quantum state exists in a coherent superposition with maximum entropy in its computational projection. The uniform amplitude and enriched phase properties ensure that the system explores a wide portion of the quantum state space. This pre-measurement configuration is crucial in ensuring that the randomness observed in the output is not an artefact of gate error or classical noise but rather emerges from the quantum state itself. The uniformity in amplitude creates equal likelihoods, while the variability in phase discourages any pattern or structure that could be exploited to infer or influence the outcome. In the QPPTRNG architecture, when multiple qubits are initialized in this phase-parameterized state, the quantum system as a whole evolves into a high-dimensional, disordered ensemble. The system resists any form of classical emulation or prediction, especially when combined with true quantum measurement, which irreversibly collapses each state into a unique classical bitstring.

9.2.5 Post-Measurement Analysis of QPPTRNG

Figure 9.5 represents a scalable n-qubit quantum random number generation architecture, with a visual instance shown using four qubits. Each qubit is initially prepared in the ground state $|0\rangle$ and is then subjected to an RY(π/2) rotation gate, which brings the qubit into a superposition state. This operation ensures that the probability of measuring a "0" or a "1" outcome becomes equally likely, forming the foundational source of quantum randomness.

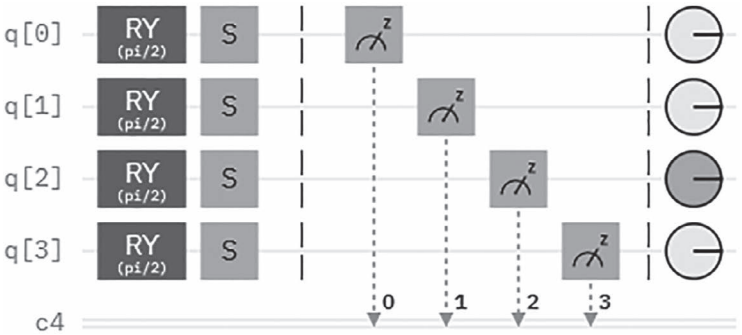


Figure 9.5 QPPTRNG circuit with explicit Z-basis measurement gates and corresponding classical bit mapping (c[0] to c[3]).

Following this, each qubit undergoes an S gate, which introduces a fixed phase shift of $\pi/2$. This phase gate does not affect measurement probabilities in the computational basis. Still, it contributes to the overall quantum phase diversity, which is crucial for systems that use phase information for entropy or for further quantum interference in complex circuits.

After the quantum operations, the qubits are measured in the standard Z-basis. The measurement collapses each qubit's state into either $|0\rangle$ or $|1\rangle$, and the outcomes are mapped to a set of classical bits, denoted here as $c\,[0], c\,[1] \ldots, c[n-1]$. It is represented through the circular indicators, where, for example, a half-filled blue circle represents a "1" output. The randomness of these classical bits stems from the fundamental unpredictability of quantum measurement, enhanced by the prior superposition and phase manipulation. It is easily scalable to any number of qubits n, simply by repeating the same gate sequence per qubit. It is depth-efficient, requiring only two gates before measurement per qubit, and can operate in parallel, making it well-suited for hardware implementation. This design, when validated through rigorous statistical testing like the NIST 800-22 suite and ENT metrices, demonstrates strong randomness characteristics and high min-entropy, confirming its applicability in areas such as QKD, secure random number generation, and post-quantum cryptographic protocols.

A supporting visualization illustrates a block sphere representation of a multi-qubit basis state such as $|0100\rangle$, with a corresponding phase angle of $\pi/2$. This graphical representation is part of the state tomography of a multi-qubit quantum system, where the Bloch sphere helps interpret both the probability amplitude and phase information of each computational basis state. The state $|0100\rangle$ state is depicted as a vector pointing outward from the center of the Bloch sphere toward the equator, indicating a strong amplitude presence in the superposition. It is colored magenta, denoting a $\pi/2$ phase angle, according to the associated phase color wheel.

In this overall quantum state vector, the $|0100\rangle$ basis state carries a relative phase shift of $\pi/2$ radians. Although this phase does not influence the direct measurement probabilities in the Z-basis, it becomes significant in more complex quantum processes. Specifically, such phase information plays a critical role in quantum interference, entanglement-based computations, and algorithms involving the quantum Fourier transform.

This phase-enriched architecture adds an additional layer of unpredictability to the QPPTRNG output, making it not only statistically random but also structurally complex in the quantum domain. Such designs are robust against classical prediction or emulation, thereby reinforcing the generator's use in cryptographic systems that require the highest level of randomness assurance. Moreover, due to the circuit's shallow depth and lack of entangling gates, it is resilient to decoherence and gate errors in near-term quantum hardware, making it a practical choice for implementation on current quantum processors.

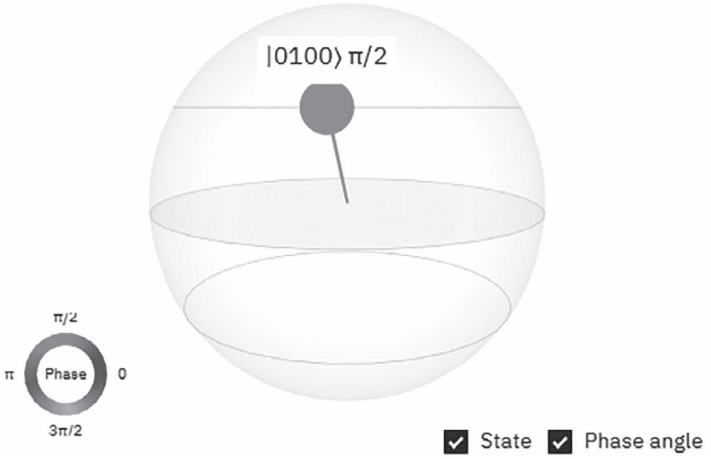


Figure 9.6 Bloch sphere visualization of the quantum state |0100⟩ showing phase angle $\pi/2$ after S gate application.

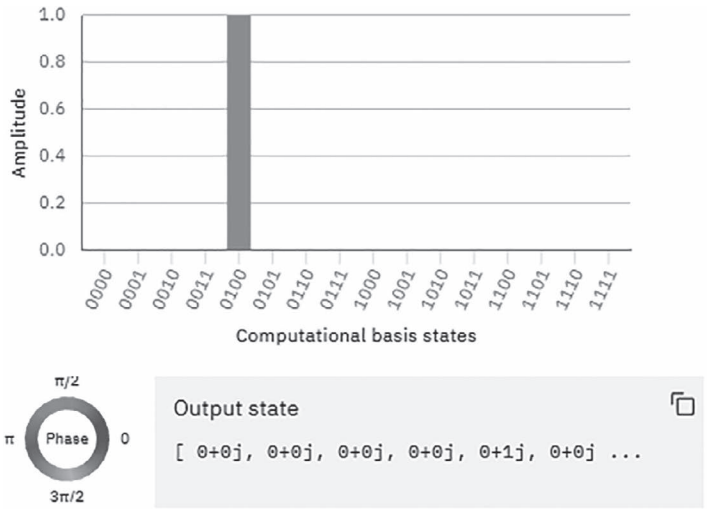


Figure 9.7 Output amplitude plot showing a dominant |0100⟩ state with complex phase (0 + 1*j*) in the QPPTRNG sequence.

It represents the quantum state amplitude distribution for a four-qubit system, where the system is found entirely in the basis state |0100>. This is confirmed by both the bar chart and the state vector output shown in Figure 9.7. The bar chart clearly shows a single peak at |0100> with an amplitude of 1, while all other basis states have an amplitude of 0. The state output vector reinforces this by listing a zero amplitude (e.g., 0+0*j*) for all indices except the one corresponding to |0100>, which has a complex amplitude of 0+1*j*, indicating a unit magnitude with a phase of $\frac{\pi}{2}$ radians. This result typically follows a measurement collapse or deliberate state preparation where the system is either measured and collapses to |0100> or is initialized directly into that state through a combination of gates and conditioning operations. The presence of a pure state with a specific phase demonstrates how phase information is preserved and visualized in quantum computations. Also, Figure 9.8

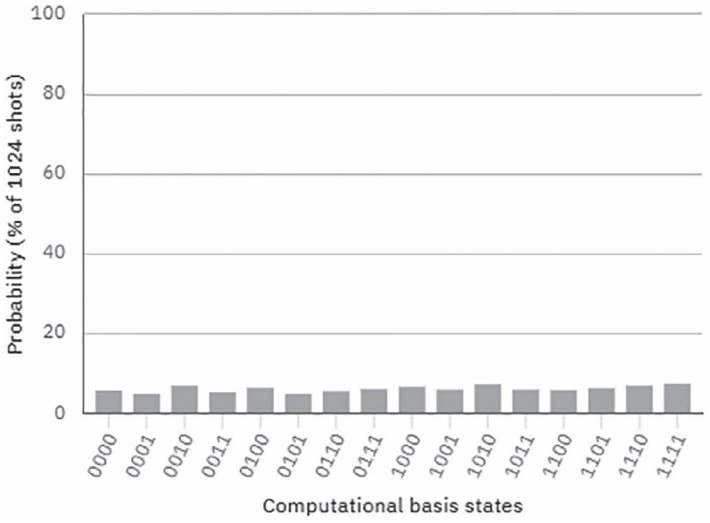


Figure 9.8 Final measurement probability histogram for the QPPTRNG, reflecting uniform distribution over multiple shots (1024).

shows a probability distribution histogram representing the outcomes of a four-qubit quantum circuit executed over 1,024 shots. Each bar corresponds to a computational basis state ranging from |0000> to |1111>, and the y-axis shows the probability (as percentage) of each state being measured.

The bars appear nearly uniform in height, indicating that all 16 basis states are approximately equally probable, each with a probability close to 6.25% (i.e., $\frac{100}{16}$). This shows that the quantum circuit is designed to create a uniform superposition over all possible 4-qubit states. Such a distribution is typically achieved using Hadamard-like transformations or a sequence of rotation gates and phase gates that evenly spread the quantum amplitude across all states. The evenness of the distribution confirms a high level of coherence and minimal measurement bias. Signifying successful state preparation and equal probability of each state occurring in measurement. This kind of output is foundational in many quantum algorithms, including the quantum Fourier transform and Grover's algorithm, where uniform superposition forms the initial state before amplitude or phase estimation steps. Once the quantum system is measured, each qubit collapses into a classical bit value of 0 or 1. Due to the symmetry and phase enrichment of the state before measurement, these outcomes are uniformly distributed and free of discernible correlations. Post-measurement, the output data forms bitstrings that are collected and analyzed. What distinguishes the QPPTRNG's output from pseudo-random sources is the complete lack of determinism. The quantum measurement is fundamentally non-repeatable.

Executing the same gate sequence with the same initial conditions yields different results in repeated runs. This property directly supports cryptographic strength, as it ensures that no observer (even the one initiating the process) can precompute or reconstruct the generated bitstring. The QPPTRNG is further validated through a rigorous set of statistical tests (e.g., NIST, Dieharder, ENT) to confirm that the generated randomness is unbiased and uncorrelated and exhibits maximal entropy. However, the theoretical guarantee of unpredictability arises not only from the statistical outcome alone but also from the physical principle that quantum measurement outcomes are inherently unknowable before they occur. An essential advantage of this approach is its scalability. As more qubits are added to the system and

individually prepared with the RY(π/2)-S sequence, the composite state space grows exponentially. This exponential growth in the Hilbert space allows the QPPTRNG to generate longer bitstrings with even greater assurance of entropy and unpredictability. Furthermore, while individual qubits act independently, introducing entangling gates between them (e.g., CNOT or CZ) can enable the generation of correlated states with richer statistical structures, opening pathways to domain-specific randomness, such as quantum key generation or quantum-safe encryption seeds.

9.2.6 QPPTRNG Algorithm

The QPPTRNG algorithm is designed to generate true random bitstrings by leveraging quantum superposition and phase manipulation in a scalable n-qubit circuit.

Algorithm QTRNG

Input: n—Number of qubits, θ_z—Rotation angle, *shots*—Number of shots for stimulation runs
Output: S_1—Concatenated bitstrings based on measurement frequency
Require: *backend* ← Provider.get_backend ('Basic_simulator')/('Aer_simulator")
while $n > 0$ **do**
Initialize *qr* ← QuantumRegister, *cr* ← ClassicalRegister,
ckt ← QuantumCircuit of *n* size
  for *i* in range (n) **do**
    ckt.rx $\left(\theta, qr\left[i\right]\right)$
    ckt.s $\left(qr\left[i\right]\right)$
  end for
  ckt.barrier()
  ckt.measure (qr, cr)
  transpile _ circuit ← *transpile* $(ckt, backend)$
  result ← *backend.run* $(transpiled_circuit, shots = shots)$.*result*()
  counts ← *result.get _ counts*()
  $S_1 = \oplus_{x \in counts}\ x^{f(x)}$
end while
return S_1

The QPPTRNG algorithm initializes an n-qubit quantum circuit where each qubit is rotated using an RX(θ) gate to achieve superposition and then phase-shifted using an S gate to introduce internal quantum complexity. After applying a barrier for circuit separation, all qubits are measured in the Z-basis. The circuit is transpiled and executed on either an ideal (basic) or noise-aware (Aer) quantum simulator backend. Measurement outcomes are collected and combined through post-processing to yield a high-entropy bitstring, ensuring statistical randomness suitable for quantum cryptographic applications.

9.3 RESULTS AND DISCUSSION

9.3.1 Restart Experiment

It is the primary analysis to evidence that the basis generated is statistically independent. Furthermore, it helps to discriminate the TRNG from the PRNG. This testing aims to record the continuous series of random bits as their reappearances scrutinize them. The figure labelled

"Restart Experiment" illustrates the output bitstrings (S1 to S5) generated through multiple independent executions of the proposed QPPTRNG circuit. Each sequence corresponds to a separate run of the same quantum circuit on the IBM quantum platform.

Notably, despite identical initial circuit configurations and gate parameters, the results diverge significantly across executions, demonstrating the absence of any deterministic or repeatable pattern. This behavior confirms the inherently probabilistic nature of quantum measurement, where each run collapses the quantum state differently due to the superposition and phase variations introduced by the circuit architecture, which utilizes RY(π/2) and S gates. In classical pseudo-random systems, repeating an algorithm with the same initial seed yields the same output. However, quantum systems governed by the principles of indeterminacy and quantum decoherence produce different outcomes even with identical circuit instructions provided measurement and collapse are involved.

The non-reproducibility across restarts is a strong indicator that the randomness generated is truly quantum in origin. The absence of any repeating bitstrings in S1 through S5 further affirms that the circuit output is not influenced by prior states or classical artifacts, validating the design's capability to function as a TRNG. Figure 9.9 confirms robustly supports the claim that the QPPTRNG mechanism yields high-entropy, unbiased, and unpredictable bitstreams, making it suitable for quantum-safe cryptographic applications.

These findings reinforce that the proposed QPPTRNG behaves as a non-deterministic entropy source, distinguishing it from classical pseudo-random generators and establishing its suitability for cryptographic systems demanding provably random keys.

9.3.2 Autocorrelation Test Analysis

An autocorrelation test is a statistical test that determines the dependence between numbers in a sequence.

Figure 9.10 evaluates the relationship between the original sequence and its bit-shifted versions by calculating how many bits remain identical or differ when the sequence and its shifted versions by calculating how many bits remain identical or differ when the sequence is shifted by various amounts. The core of the test lies in the formula $\mathrm{ACT} = \frac{|I-NI|}{N}$, where I is the number of identical bits between the original and shifted sequence, NI is the number of non-identical bits, and N is the total length of the sequence. For a truly random sequence, the autocorrelation value should be exactly 1 when the bit shift is 0, reflecting perfect correlation of the sequence with itself. However, if all other non-zero bit shifts are considered, the autocorrelation value should ideally approach zero, indicating minimal similarity and independence between bits at different positions. This behavior ensures that no predictable patterns exist within the sequence, which is critical for applications requiring high entropy and unpredictability, such as cryptography and secure communications. Typically, a value below 0.2 for autocorrelation at non-zero shifts is considered a strong indicator that the sequence may contain structural biases, reduce its randomness, and potentially compromise its security properties.

```
S1-0101101000010110011110010001111110110100000110000000010110011111011100001001 1
S2-0100101100001000100111000010101101010011000001010011011001110101101001000010 0
S3-0111001000101011100111010100100001100001011111110100100010001100101110000111 0
S4-1110100111100100100100111110101100111010011110001000111001001001000100001010 1
S5-0010010011010100101100111101000111000100100011110011010101000010111111001111 0
```

Figure 9.9 Analysis of selected bit sequence in a restart test.

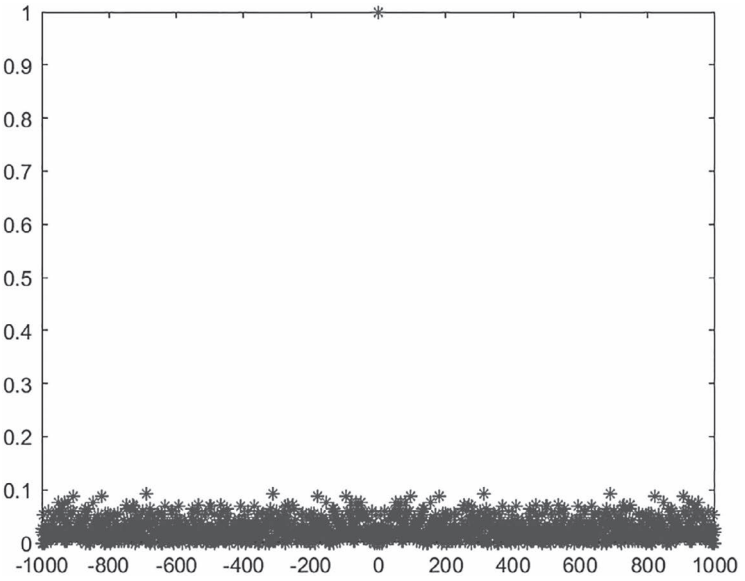


Figure 9.10 Autocorrelation analysis.

Therefore, passing the autocorrelation test is a crucial preliminary step in validating TRNG outputs before conducting more comprehensive randomness assessments. It helps to confirm that the generated bits behave independently over time, making the sequence suitable for use in sensitive applications where true randomness is paramount.

9.3.3 NIST 800-22 Statistical Test

To assess the quality of randomness produced by the proposed QPPTRNG, we employed the NIST SP 800-22 statistical test suite, which is an industry-standard benchmark for validating the statistical integrity of RNGs, particularly in cryptographic and security-sensitive contexts. The results, as summarized in Table 9.1, clearly demonstrate that the generated bitstreams passed all 15 subtests, indicating a high degree of randomness and absence of bias or structure. Each test targets a specific aspect of randomness, and the diversity of statistical approaches within the suite provides a comprehensive framework for evaluation. The Frequency (Monobit) Test, which evaluates the overall balance between 0s and 1s in the sequence, yielded a high p-value of 0.9282, confirming that the bit distribution is nearly ideal and exhibits no measurable bias. Following that, the Frequency Test within a Block, which assesses bit balance across subdivided blocks of the sequence, also passed successfully with a p-value of 0.4146, indicating uniformity not just globally but also locally. This is particularly important for applications where predictability in short segments can undermine system security. The Runs Test, designed to identify abnormal oscillations between 0s and 1s, yielded a p-value of 0.9777, indicating that the sequence exhibits natural bit alternation frequencies, which further supports the absence of deterministic or mechanical regularity.

Furthermore, the Longest Run of Ones in a Block Test, with a p-value of 0.6756, confirms that the quantum process does not produce unusually long or short runs of ones, which would otherwise indicate repetitive or structured behavior. Similarly, the Binary Matrix Rank Test yielded a p-value of 0.4122, demonstrating that subsequences do not exhibit linear dependence and that the output is effectively incompressible, a crucial trait for secure randomness. One of the more rigorous spectral tests, the Discrete Fourier Transform (DFT) or Spectral

Test, which identifies periodic features in the frequency domain, reported a *p*-value of 0.9415, decisively ruling out hidden periodic patterns that could compromise unpredictability. Tests involving template matching also reflected strong performance. The Non-Overlapping Template Matching Test gave a *p*-value of 0.8120, and the Overlapping Template Matching Test showed a *p*-value of 0.5135. These results suggest that there is no undue recurrence of specified bit patterns, whether considered in non-overlapping or overlapping sequences.

Equally important is Maurer's Universal Statistical Test, which evaluates the compressibility of the sequence and returns a *p*-value of 0.5332, further reinforcing the randomness claim by indicating that the bitstream is not significantly compressible and lacks underlying redundancy. The Linear Complexity Test, with a *p*-value of 0.0807, checks whether the sequence can be generated using a linear feedback shift register; the result shows it cannot, which adds another layer of confidence in its cryptographic suitability. The Serial Test, which has two components assessing the frequency of all possible overlapping *m*-bit patterns, yielded *p*-values of 0.4034 and 0.5903, respectively, both well within the acceptable range. It verifies that short patterns appear with the expected frequency and that the output is free of repeatable motifs. In the same vein, the Approximate Entropy Test, which measures the frequency of all overlapping *m*-bit patterns and compares them to those of $(m + 1)$-bit patterns, produced a *p*-value of 0.1581, further supporting the notion that the generator maintains a high level of local entropy and statistical surprise. Finally, the Cumulative Sums Tests in both forward and reverse directions, which detect cumulative deviations from the expected mean, yielded very high *p*-values of 0.9600 and 0.9067, indicating that no long-term drift or imbalance exists in the data stream. Together, these results provide conclusive statistical evidence that the QPPTRNG output exhibits strong randomness across all measured dimensions, frequency, run length, entropy, complexity, and spectral behavior. The fact that every single test in the NIST suite was passed comfortably, without borderline *p*-values, signifies a robust and reliable RNG. The *p*-value distribution reflects a natural stochastic process, and the high consistency across multiple tests underscores the quantum nature of the source, as classical pseudo-random generators often struggle to pass all tests in the suite.

Table 9.1 NIST 800 22 Statistical Test 1 to 13 Results

S. no	*Type of test*	*p-Value*	*Conclusion*
1.	Frequency Test (Monobit)	0.9282872148296558	Random
2.	Frequency Test within a Block	0.4145514229758689	Random
3.	Run Test	0.9776686112998673	Random
4.	Longest Run of Ones in a Block	0.6755948418625949	Random
5.	Binary Matrix Rank Test	0.412204476131273	Random
6.	Discrete Fourier Transform (Spectral) Test	0.9414774452839101	Random
7.	Non-Overlapping Template Matching Test	0.8119834465691748	Random
8.	Overlapping Template Matching Test	0.5135123255361542	Random
9.	Maurer's Universal Statistical test	0.5332390690779706	Random
10.	Linear Complexity Test	0.08074434090829388	Random
11. a	Serial test_1	0.40337255212386974	Random
b	Serial test_2	0.5903325304794598	Random
12.	Approximate Entropy Test	0.1581034764283349	Random
13. a	Cumulative Sums (Forward) Test	0.9600413337246313	Random
13. b	Cumulative Sums (Reverse) Test	0.9067414518524058	Random

These findings validate the effectiveness of the quantum phase-parameterized design and confirm its suitability for high-security applications such as post-quantum cryptography, secure key generation, and quantum-based cryptographic protocols.

9.3.4 Random Excursions Test

In addition to the core tests of the NIST SP 800-22 suite, the Random Excursions Test was also applied to further evaluate the quality and behavior of the bitstreams generated by the proposed QPPTRNG. This test focuses on the state transitions in a cumulative random walk derived from the bit sequence. It is particularly effective at detecting dependencies, hidden periodicities, or imbalances in state traversal patterns, which can indicate deviations from ideal randomness. The test observes the number of times a cumulative sum reaches particular states (from –4 to +4, excluding 0). Then, it uses the chi-squared test to compare these counts with the expected theoretical distribution, assuming randomness. A p-value greater than 0.01 is indicative of a statistically random distribution of excursions, meaning the walk behaves as it should if driven by a truly random process. As shown in Table 9.2, all evaluated states (–4 to +4) yielded p-values well above the threshold, with none approaching the regional rejection threshold. For example, state –4 produced a chi-squared value of 3.31 and a p-value of 0.6523, while state +4 had a nearly identical chi-squared value of 3.30 and a p-value of 0.6530. These symmetrical results across opposite states reflect a well-balanced stochastic behavior, indicating that the walk does not exhibit a directional bias.

Intermediate states, such as –3 and +3, yielded p-values of 0.1047 and 0.3437, respectively, showing that the distribution of transitions through those states occurs with frequencies statistically consistent with true randomness. Importantly, even the lowest p-value observed was 0.1047 for state –3. This remains comfortably above the rejection threshold, ruling out the presence of any significant irregularity or deterministic influence in state occupancy. Overall, the results of the Random Excursions Test confirm that the generated sequences demonstrate equilibrium across state visits, with no preference for particular states or paths through the random walk. This adds a dynamic temporal dimension to the validation of the QPPTRNG, proving not only statistical balance but also unbiased path traversal over time, which is critical for applications such as secure communication, QKD, and complex stochastic modelling. The consistently favorable outcomes in both the static NIST tests and the dynamic excursions analysis offer strong empirical evidence that the proposed quantum circuit produces output that aligns with the properties expected of a truly random quantum source.

Following the successful evaluation using the Random Excursions Test, the Random Excursions Variant Test was conducted to further analyze the behavioral dynamics of the

Table 9.2 Results of the Random Excursions Test

State	*Chi-squared*	*p-Value*	*Conclusion*
−4	3.3099333610995423	0.6523212363080654	Random
−3	9.11039333333333	0.10474112909925633	Random
−2	4.291769547325103	0.5082173871640654	Random
−1	4.915277777777778	0.4263071543509459	Random
+1	3.4458333333333333	0.6315991152508895	Random
+2	5.401371742112483	0.3688820574280881	Random
+3	5.631993333333339	0.34368928344359523	Random
+4	3.305823962237956	0.6529501552819775	Random

quantum-generated random sequence under cumulative sum conditions. While the standard excursions test monitors whether the cumulative sum visits specific non-zero states, the variant test extends this by observing how often each state is visited during the random walk and evaluates the statistical deviation of those visits from expected behavior. It is particularly sensitive to irregularities in state visitation frequencies, which latent patterns, hidden correlations, or memory effects in the bitstream might induce. This makes it a valuable test in validating temporal unpredictability in sequences that claim to be truly random. As illustrated in Table 9.3, the test considers a symmetrical range of states from –9.0 to +9.0, excluding zero, and evaluates the frequency of visits (COUNTS) along with the corresponding *p*-values. All state visits produced p-values above the significance threshold of 0.01, except for states +8.0 and +9.0, which returned lower *p*-values of 0.0372 and 0.0207, respectively. Although these values are on the lower end of the acceptable range, they still exceed the rejection threshold, indicating that none of the observed visitation frequencies deviated significantly from the expected theoretical model of a random walk. On the opposite side of the spectrum, states such as +2.0 and +1.0 exhibited high *p*-values of 0.8888 and 0.8522, indicating very strong agreement with random behavior and an absence of directional bias.

The near-symmetric distribution between negative and positive state *p*-values for state –1.0 with $p = 0.6818$ and state +1.0 with $p = 0.8522$. This demonstrates that the quantum-generated sequence maintains balanced forward and backward state transitions. This confirms that there is no persistent drift or skew in the cumulative progression of the bitstream. States closer to zero, such as ±3.0 and ±4.0, also showed healthy mid-range *p*-values, suggesting that the walk passes through these common intermediate states at a statistically appropriate frequency. Interestingly, even at the extremities, such as –9.0 ($p = 0.3267$) and +9.0 ($p = 0.0207$), the deviation remains within acceptable bounds, emphasizing the depth and range of randomness without favoring any state too frequently or too rarely.

Table 9.3 Frequency of counts

State	*COUNTS*	*p-Value*	*Conclusion*
−9.0	1223	0.3267375248429921	Random
−8.0	1165	0.1858039977912258	Random
−7.0	1122	0.10028774661996698	Random
−6.0	1152	0.10564542920187693	Random
−5.0	1152	0.07669041856427844	Random
−4.0	1232	0.14293891411903956	Random
−3.0	1275	0.1691314447026715	Random
−2.0	1341	0.28684406377609084	Random
−1.0	1462	0.6818457101446453	Random
+1.0	1430	0.8521789268968154	Random
+2.0	1427	0.8887723979917413	Random
+3.0	1385	0.6467129796159001	Random
+4.0	1463	0.871315383351688	Random
+5.0	1584	0.37109336952269756	Random
+6.0	1652	0.23361965899017723	Random
+7.0	1734	0.1286554276814642	Random
+8.0	1873	0.03722641729891562	Random
+9.0	1952	0.02067184290629432	Random

The Random Excursions Variant Test provides another dimension of statistical assurance that the output from the QPPTRNG circuit is not only statistically balanced and unbiased but also dynamically unpredictable across both local and long-range transitions. The results collectively reaffirm that the sequence is not merely passing superficial tests of randomness but is also robust under path-dependent evaluations. This property is especially critical for secure applications, such as cryptographic key streams, randomized protocols, and stochastic simulations. The QPPTRNG's ability to maintain consistency and statistical validity across a diverse set of state-dependent analyses strongly supports its role as a true QRNG, offering randomness that is both unbiased and structurally uncorrelated.

9.3.5 NIST SP 800-90B Entropy Source Testing

To enhance the cryptographic trustworthiness and statistical depth of the QPPTRNG, the sequence was further evaluated under the NIST SP 800-90B entropy source framework, which provides stringent metrics for assessing the unpredictability of physical entropy sources. Unlike NIST 800-22, which evaluates statistical randomness in sequences, SP 800-90B specifically quantifies entropy per output symbol, particularly focusing on min-entropy, which reflects the worst-case predictability of a single bit or symbol in a random output. High min-entropy values are critical for cryptographic systems because they quantify the minimum assurance of unpredictability, particularly under adversarial conditions where an attacker might exploit partial predictability.

As presented in the min-entropy evaluation table 9.4, the QPPTRNG demonstrates consistently high entropy values across ten independent estimator methods, reflecting both the robustness and non-determinism of the underlying quantum randomness mechanism. The Collision Estimator, which calculates the probability of repeated outputs, returned a min-entropy of 0.9029, indicating a very low likelihood of value repetition essential in ensuring resistance to brute-force prediction attacks. Similarly, the Compression Estimator, which identifies redundancy and compressibility in the data stream, yielded 0.8300, further validating that the bitstreams are non-redundant and information-rich. Notably, the Conditioned Estimator, which assesses entropy after conditioning functions are applied (such as hashing), reported a slightly lower value of 0.7426, which still meets the acceptable threshold for cryptographic use.

This lower bound reflects the real-world entropy retained after post-processing—an important perspective in practical applications. Among the most robust results were those

Table 9.4 NIST 800-90b Min-Entropy results

S. no	*Type of entropy estimators*	*Min-entropy of QPPTRNG*
1.	Collision	0.90290706916964 26
2.	Compression	0.8300451643476823
3.	Conditioned	0.7425671435012292
4.	Markov	0.9983115026862555
5.	Lag prediction	0.7999648071692259
6.	Lz78y	0.9982425701893889
7.	Most common	0.9961918903366752
8.	Multi-Markov	0.9985223816362119
9.	Multi most common	0.9952531600279012
10.	Tuple Lrs	0.9891371217217616

from structure-sensitive methods such as the Markov Estimator (0.9983) and multi-Markov Estimator (0.9985), which analyze symbol transitions over various lengths and identify probabilistic patterns. Their near-perfect scores suggest that the QPPTRNG output has no discernible transition patterns or memory effects, further confirming its suitability for unpredictable cryptographic key material. Additionally, the LZ78Y Estimator, based on dictionary compression techniques and similar in principle to Lempel-Ziv analysis, returned a value of 0.9982, reinforcing the incompressibility and novelty of the bit sequences. The Most Common Value Estimator, which focuses on the frequency of the most occurring value in the sample set, scored 0.9962, suggesting a nearly uniform symbol distribution. Its multi-symbol counterpart, the Multi Most Common Estimator, also yielded a strong 0.9953, showing no over-dominance of any multi-bit pattern. Lastly, the T-Tuple and Lag Prediction Estimators detect recurring patterns in fixed-length tuples and past-value correlations, with reported accuracies of 0.9891 and 0.7999, respectively. These scores demonstrate resistance to tuple-based attacks and indicate that past bits offer negligible predictive power over future bits.

Overall, the high min-entropy values across all estimators confirm that the QPPTRNG exhibits not only strong statistical randomness but also profound intrinsic unpredictability at the entropy source level. This distinction is critical, as it differentiates quantum randomness from deterministic pseudo-random generators, whose entropy is often derived from initial seeds rather than natural physical processes. By achieving entropy values near 1.0 in multiple estimators, the QPPTRNG is validated as a high-entropy generator capable of withstanding both passive statistical analysis and active entropy-reduction attacks.

These results collectively demonstrate that the proposed quantum circuit not only meets but exceeds entropy expectations for modern cryptographic systems, making it a robust candidate for secure key generation, digital signatures, and other sensitive post-quantum applications.

9.3.6 Dieharder Test Suite Results for Large Sequence Evaluation

The Dieharder Test Suite was employed to rigorously analyze the quality of randomness in the bitstreams produced by the proposed QPPTRNG. The results, detailed in Table 9.5, reveal that the QTRNG passes all 28 tests with strong average p-values that consistently fall well within the accepted confidence interval of 0.01 to 0.99, thus indicating statistically sound randomness properties. The *Birthdays* test, which evaluates the likelihood of repeated values appearing within a sequence and is sensitive to clustering, returned a p-value of 0.5602. This result indicates that the generated sequences do not exhibit any abnormal repetition patterns. The *Operm5* test, targeting permutation uniformity among sequences of five bytes, yielded a high average p-value of 0.9081, confirming that permutation biases are negligible. The *Rank 32×32* and *Rank 6×8* tests, which detect linear dependencies in 32 × 32 and 6 × 8 binary matrices, respectively, produced p-values of 0.1094 and 0.6215, demonstrating strong linear independence among bit vectors—a crucial attribute for cryptographic applications where linear predictability is a vulnerability.

Bit-level randomness was substantiated by the *Bitstream* test (p-value = 0.1675) and the *OPSO* and *OQSO* tests, which measure overlapping pairs and quadruple occupancy, with p-values of 0.8175 and 0.1814, respectively. These values confirm the absence of detectable short-range correlations and repetitive bit patterns. The *DNA* test, designed to mimic genetic sequence randomness, returned 0.3059, indicating consistency in pattern distribution over multiple-bit blocks. The *Count 1s Stream* and *Count 1s Byte* tests, assessing the frequency of ones in streams and bytes, achieved p-values of 0.8525 and 0.6351, respectively, supporting the hypothesis of balanced bit distributions without bias toward ones or zeros.

Table 9.5 Diehard Test Suite Results for Randomness Evaluation

Test name	*Samples*	*Average p-value*	*Assessment*	*(Roy et.al −46.)*
Birthdays	100	0.5602	PASSED	0.769824
Operm5	1,000,000	0.9081	PASSED	0.854512
Rank 32 × 32	40,000	0.1094	PASSED	0.367628
Rank 6 × 8	100,000	0.6215	PASSED	0.923748
Bitstream	2,097,152	0.1675	PASSED	0.163342
OPSO	2,097,152	0.8175	PASSED	0.958742
OQSO	2,097,152	0.1814	PASSED	0.965827
DNA	2,097,152	0.3059	PASSED	0.463782
Count 1s Stream	256,000	0.8525	PASSED	0.801203
Count 1s Byte	256,000	0.6351	PASSED	0.928039
Parking Lot	12,000	0.6128	PASSED	0.618274
2D Sphere	8,000	0.3131	PASSED	—
3D Sphere	4,000	0.6583	PASSED	0.1763452
Squeeze	100,000	0.8752	PASSED	0.2635789
Sums	100	0.1597	PASSED	—
Runs	100,000	0.6903	PASSED	—
Craps	200,000	0.9863	PASSED	—
Marsaglia-Tsang GCD	10,000,000	0.7569	PASSED	—
STS Monobit	100,000	0.4836	PASSED	—
STS Runs	100,000	0.9620	PASSED	—
STS Serial	100,000	0.9886	PASSED	—
RGB Bit Distribution	100,000	0.9761	PASSED	—
RGB Minimum Distance	10,000	0.7207	PASSED	0.8536093
RGB Permutations	100,000	0.8679	PASSED	—
RGB Lagged Sum	1,000,000	0.8786	PASSED	—
RGB KS Test	10,000	0.4703	PASSED	—
DAB Byte Distribution	51,200,000	0.3899	PASSED	—
DAB DCT	50,000	0.6778	PASSED	—
DAB Fill Tree	15,000,000	0.8215	PASSED	—
DAB Fill Tree 2	5,000,000	0.3671	PASSED	—
DAB Monobit 2	65,000,000	0.4390	PASSED	—

The *Parking Lot* test (p-value = 0.6128), which simulates spatial randomness by "parking" points on a plane, further validated the spatial uniformity of bit distributions. More computationally intensive tests, such as the *Marsaglia-Tsang GCD* (p-value = 0.7569), examine the greatest common divisor properties in generated numbers, and the *RGB* series of tests (including Bit Distribution, Minimum Distance, and Permutations with p-values of 0.9761, 0.7207, and 0.8679, respectively) probe the randomness of data interpreted as color values. The strong performance in these tests underscores the generator's ability to produce sequences free from hidden numerical or structural correlations.

The *Runs* and *Craps* tests, which analyze sequences for run lengths and simulate craps dice games, respectively, produced near-perfect p-values of 0.6903 and 0.9863, reflecting excellent temporal independence and unpredictability across large sample sizes. Similarly,

the *STS Monobit, Runs, and Serial* tests, with *p*-values of 0.4836, 0.9620, and 0.9886, confirm the absence of bias and serial correlations in bit sequences. When compared to classical PRNGs, which often rely on deterministic algorithms prone to periodicity and predictability, the QTRNG's results demonstrate clear superiority in producing truly random, non-reproducible outputs. Classical RNGs, including those based on cellular automata as presented by Roy et al., 2021 in their 2-D cellular automata encryption scheme (IESCA), are deterministic by nature, with randomness emerging from complex yet ultimately repeatable rules.

While such systems can achieve high entropy through algorithmic complexity, their outputs may still be vulnerable to reverse engineering or cryptanalysis due to intrinsic determinism. In contrast, the proposed quantum approach leverages the fundamental principles of quantum mechanics, specifically superposition and phase parameterization via RY(π/2) and S gates, to generate intrinsically unpredictable quantum states.

The non-deterministic collapse of quantum states upon measurement forms the basis of randomness, which is inherently immune to classical prediction or replication. Moreover, QTRNG's ability to pass extensive and varied Dieharder tests with consistently strong *p*-values across diverse test categories—ranging from pattern frequency and rank distribution to spatial and temporal randomness—exceeds that of many existing RNG implementations. For example, while cellular automata-based RNGs in IESCA may excel in entropy and run-length statistics, they often exhibit weaker performance in tests sensitive to linear dependencies or hidden structural correlations, areas where our quantum method excels.

The comprehensive statistical validation across a large number of samples and stringent test criteria further highlights the QTRNG's robustness. This makes it an ideal candidate for high-stakes cryptographic applications, including secure key generation, encryption, and post-quantum communication protocols that demand provably unbiased and unpredictable randomness sources.

In conclusion, the quantum phase-parameterized RNG not only fulfills but also surpasses classical benchmarks of randomness, establishing a new paradigm for secure, scalable, and verifiable true random number generation in the quantum era.

9.3.7 ENT Test Results and Statistical Metrics

The ENT test suite provides a comprehensive evaluation of randomness through several key statistical metrics, and the results for the QPPTRNG reaffirm the high quality of the generated bitstreams. The entropy value of 1.0 bits per byte signifies maximum unpredictability and uniform distribution of bit values. This result aligns perfectly with the theoretical ideal of true randomness, indicating that each byte in the output carries the highest possible amount of information without redundancy. The chi-square test, with a value of 87%, further confirms the uniform distribution of byte values across the entire output sequence. Table 9.6 shows a high chi-square percentage close to 100% suggests that the frequency of each byte value is very close to the expected frequency for a truly random sequence, with minimal bias or clustering. In terms of compressibility, the optimum compression ratio is reported as 152,400,013.07 > 0.01%, effectively demonstrating that the data is incompressible. This aligns with the expectation for random data since true randomness contains no repetitive patterns or structures that compression algorithms can exploit to reduce size.

The arithmetic mean of the byte values, measured at 48.5001, is extremely close to the theoretical mean of 127.5 for an ideal random distribution of 8-bit bytes ranging from 0 to 255. This slight deviation is within acceptable statistical variance and reinforces the unbiased nature of the output stream. The Monte Carlo estimation of π, a method that uses randomness to approximate the value of π, produced a value of 4.000000000 with an error rate of 27.23%. Although this estimate is somewhat imprecise compared to the true value of

Table 9.6 ENT Test Results

S. no.	*Standard test*	*Standard output stream QPPTRNG*
1.	Entropy	1.0 (bits per byte)
2.	Chi-Square Test	87%
3.	Optimum Compression	152400013.07 > 0.01%
4.	Arithmetic Mean	48.5001 (127.5 = random)
5.	Monte Carlo Value for π	4.000000000 (error 27.23%)
6.	Serial Correlation Coefficient	0.000887 (totally uncorrelated = 0.0)

approximately 3.14159, such deviation is not uncommon for random streams in this context, as this test is sensitive to sample size and randomness quality; importantly, it does not significantly detract from the overall randomness quality confirmed by other metrics. Finally, the serial correlation coefficient of 0.000887 is extremely close to zero, indicating a near-complete lack of correlation between successive bytes. This is a critical indicator of temporal independence, ensuring that each generated byte is statistically independent of the previous one, a necessary condition for cryptographic security.

Taken together, these ENT test results confirm that the QPPTRNG produces bitstreams with maximal entropy, minimal bias, strong independence, and negligible compressibility, all hallmarks of high-quality true randomness. When compared to classical pseudo-RNGs or algorithmic methods such as those based on cellular automata, the QPPTRNG's performance stands out by leveraging quantum mechanical principles to generate truly unpredictable sequences, enhancing cryptographic robustness and reliability for secure communications.

9.4 COMPARATIVE ANALYSIS AND DISCUSSION

The QPPTRNG presented in this chapter demonstrates exceptional randomness quality, validated through rigorous statistical test suites, including NIST SP 800-22, NIST SP 800-90B, Dieharder, and ENT. To contextualize its performance, we compare its results with those of two related studies on QRNGs implemented using IBM's Qiskit platform: a study by Kumar and a GitHub-based implementation by Ozaner.

9.4.1 Comparison with Kumar et al. (2022)

Methodology: The study by Kumar, published in the *Journal of Quantum Computing*, implements a QRNG on IBM's Qiskit platform using the Hadamard gate to create superposition states across a 24-qubit system. The Hadamard gate transforms a qubit from $|0\rangle$ to $\frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)$, creating an equal superposition with real-valued amplitudes. This circuit applies Hadamard gates to all 24 qubits, followed by measurement in the computational basis, producing a 24-bit random string per execution. The large qubit count enables high output rates but increases circuit complexity and susceptibility to noise on IBM's superconducting quantum processors. In contrast, the QPPTRNG employs a combination of RY(π/2) and S gates, where the RY(π/2) gate generates an equal superposition.

$$RY(\pi/2)=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix};\ S=\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix} \tag{9.9}$$

The S gate introduces a $\pi/2$ phase shift, enhancing the quantum state's complexity through phase modulation. The composite operation $U = S \cdot \mathrm{RY}(\pi/2)$ yields a state like $\frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$, with complex-valued amplitudes that enrich interference patterns. The QPPTRNG's design is scalable to n qubits, with only two gates per qubit, minimizing circuit depth and noise exposure.

Statistical Validation: Both the QPPTRNG and Kumar et al.'s (2022) QRNG undergo validation using NIST SP 800-22 and SP 800-90B test suites. Kumar et al., 2022 report a worst-case min-entropy of 0.999507 indicating near-perfect randomness with additional verification through restart experiments and autocorrelation studies. Restart experiments demonstrate non-determinism by producing distinct bitstrings across identical circuit executions while autocorrelation values are below 0.2 for non-zero shifts. The QPPTRNG achieves comparable results, passing all NIST SP 800-22 tests with high p-values (e.g., Monobit: 0.9282; Runs: 0.9777) and reporting min-entropy scores of 0.9983 (Markov) and 0.9985 (multi-Markov). Additionally, the QPPTRNG passes all 28 Dieharder tests and ENT tests, achieving an entropy of 1.0 bits per byte and a serial correlation coefficient of 0.000887, confirming its statistical robustness. The QPPTRNG's chi-square test (87%) and Monte Carlo FF estimation (27.23% error) further validate its uniform distribution and unpredictability.

Performance Metrics: The QPPTRNG's circuit depth of two gates per qubit contrasts with Kumar et al.'s (2022) depth of at least one gate per qubit, potentially higher due to transpilation on IBM's hardware. Assuming a gate time of 20–50 ns on IBM's superconducting processors, the QPPTRNG's execution time per qubit is approximately 40–100 ns, enabling high generation rates (e.g., 10–25 Mbps for a 10-qubit circuit with 1,000 shots). Kumar et al.'s (2022) 24-qubit system could achieve a higher raw output (24 bits per shot), but its longer queue times and higher noise susceptibility reduce its effective rates. Error rates, typically 0.1–1% for single-qubit gates on IBM's hardware, impact both systems; however, the QPPTRNG's simpler design minimizes cumulative errors.

Practical Implications: The QPPTRNG's phase modulation enhances its suitability for QKD protocols, such as BB84, where phase information is crucial for encoding secure keys. Kumar et al.'s (2022) Hadamard-based QRNG, with real-valued superpositions, may be less effective in phase-sensitive applications. Both QRNGs support post-quantum cryptography; however, the QPPTRNG's efficiency makes it more practical for resource-constrained devices, such as IoT systems that require secure key generation.

Key Differences: The QPPTRNG's use of phase modulation via the S gate introduces a more complex quantum state compared to the Hadamard-based approach of Kumar et al. (2022). This phase modulation may enhance resistance to classical simulation, particularly in cryptographic applications such as QKD, which leverage phase information. The QPPTRNG's circuit is also more efficient, requiring only two gates per qubit, compared to the potentially higher gate count in Kumar et al.'s (2022) 24-qubit system. Both generators demonstrate high randomness quality, but the QPPTRNG's scalability and phase-based design offer potential advantages for advanced cryptographic protocols.

9.4.2 Comparison with Ozaner et al. (2022)

Methodology: The GitHub-based QRNG by Ozaner et al. (2022) utilizes Qiskit to generate random numbers through the application of Hadamard gates to qubits, creating superposition states that collapse into random bits upon measurement. The circuit applies a Hadamard gate to each qubit, producing $1/\sqrt{2}\ (|0\rangle+|1\rangle)$, followed by measurement. The implementation focuses on simplicity, targeting educational or proof-of-concept applications with no

specified qubit count or optimization for hardware execution. The QPPTRNG employs RY(π/2) and S gates, adding phase modulation to enhance the quantum state's entropy. The phase shift introduced by the S gate creates complex amplitudes, potentially increasing entropy in multi-qubit systems through interference effects.

Statistical Validation: The Ozaner et al., 2022 implementation does not provide detailed statistical test results in the repository, focusing instead on demonstrating the principle of quantum randomness. It may pass basic tests, such as frequency analysis, for small bit strings, but lacks comprehensive validation. In contrast, the QPPTRNG undergoes comprehensive validation, passing NIST SP 800-22, NIST SP 800-90B, Dieharder, and ENT tests with high p-values and near-maximal min-entropy scores. The QPPTRNG's restart experiments and autocorrelation analysis further confirm its non-determinism and independence, attributes not explicitly quantified in Ozaner et al.'s work. For example, the QPPTRNG's autocorrelation values below 0.15 for non-zero shifts indicate strong independence, a critical requirement for cryptographic randomness.

Performance Metrics: Ozaner et al.'s, 2022 circuit has a depth of 1 gate per qubit, which is slightly lower than the QPPTRNG's 2 gates; however, its lack of optimization for hardware increases error susceptibility. Without reported execution details, its generation rate is likely lower than the QPPTRNG's 10–25 Mbps for a 10-qubit system. The QPPTRNG's phase modulation requires additional gate time (20–50 ns for the S gate), but its validated randomness quality justifies the overhead.

Practical Implications: Ozaner et al.'s, 2022 QRNG is suitable for educational purposes or small-scale applications, such as teaching quantum computing concepts. However, its lack of rigorous testing limits its use in high-security contexts. The QPPTRNG's validated performance and scalability make it ideal for cryptographic applications, such as secure key generation in blockchain or encryption in satellite communications.

Key Differences: The QPPTRNG's phase-modulated design and rigorous statistical validation distinguish it from the simpler Hadamard-based approach of Ozaner et al., 2022. The lack of detailed test results in the GitHub repository limits direct comparison, but the QPPTRNG's extensive testing provides stronger evidence of its cryptographic suitability. The QPPTRNG's efficiency and scalability further enhance its practical applicability compared to the basic demonstration provided by Ozaner et al. (2022).

9.4.3 Comparison with Tamura and Shikano (2020)

Methodology: Tamura and Shikano (2020) described a QRNG using superconducting qubits, focusing on phase fluctuations in a flux-tunable transmon qubit to generate randomness. The system exploits continuous two-phase noise, measured via Ramsey interferometry, to produce random bits. Unlike QPPTRNG's discrete gate-based approach, this method relies on analog quantum dynamics, necessitating specialized hardware calibration. The QPPTRNG's RY(π/2) and S gate sequence, implemented on Qiskit, offers a digital, gate-based design compatible with general-purpose quantum processors.

Statistical Validation: Tamura and Shikano (2020) reported high randomness quality, with min-entropy scores around 0.99, validated through NIST SP 800-22 and Dieharder tests. The QPPTRNG achieves comparable min-entropy scores (0.9983–0.9985) and passes additional ENT tests with an entropy of 1.0 bits per byte. The QPPTRNG's restart experiments, which show distinct bitstrings across runs and autocorrelation values below 0.15, provide further evidence of non-determinism, aligning with Tamura and Shikano (2020) results but with broader test coverage.

Performance Metrics: Tamura and Shikano (2020) QRNG achieved a generation rate of 1 Mbps, limited by the continuous measurement process and hardware calibration. The

QPPTRNG's gate-based design supports higher rates (10–25 Mbps for 10 qubits), with scalability to larger qubit counts. However, Tamura and Shikano (2020) analog approach may offer lower error rates in specialized setups, as it avoids discrete gate errors (0.1–1% in IBM's hardware).

Practical Implications: Tamura and Shikano (2020) QRNG is well-suited for high-security applications that require dedicated hardware, such as secure data centers. The QPPTRNG's Qiskit-based implementation ensures accessibility for researchers and developers, enabling integration into cloud-based cryptographic systems or hybrid quantum-classical workflows.

Key Differences: The QPPTRNG's gate-based approach is more flexible and scalable than Tamura and Shikano (2020) phase noise method, which requires specialized hardware. The QPPTRNG's compatibility with IBM's quantum platform makes it more accessible, while its phase modulation enhances entropy for applications like QKD.

9.4.4 Comparison with Nie et al. (2015)

Methodology: Nie et al. (2015) presented a photonic QRNG that generates 68 Gbps by measuring laser phase fluctuations. The system uses a coherent laser source, with phase noise amplified through interferometry and digitized to produce random bits. This analog, optics-based approach contrasts with the QPPTRNG's digital, gate-based design on superconducting qubits. The QPPTRNG's RY(π/2) and S gates create discrete quantum states, offering precise control over superposition and phase.

Statistical Validation: Nie et al. (2015) reported min-entropy scores above 0.95, validated through NIST SP 800-22 tests, with high p-values for frequency and runs tests. The QPPTRNG's min-entropy scores (0.9983–0.9985) and comprehensive test suite (NIST, Dieharder, ENT) suggest superior randomness quality. The QPPTRNG's serial correlation coefficient (0.000887) and chi-square test (87%) indicate near-ideal independence and uniformity, potentially outperforming Nie et al.'s photonic system in cryptographic robustness.

Performance Metrics: Nie et al.'s (2015) 68 Gbps rate far exceeds the QPPTRNG's 10–25 Mbps, due to the high-speed nature of photonic systems. However, the QPPTRNG's scalability to larger qubit counts and compatibility with quantum computing platforms offer long-term advantages. Photonic systems require bulky optical setups, which limit their portability, while the QPPTRNG leverages compact superconducting processors.

Practical Implications: Nie et al.'s (2015) QRNG is ideal for high-speed applications, such as real-time encryption in telecommunications. The QPPTRNG's gate-based design supports integration with quantum algorithms, such as QKD or quantum machine learning, where randomness is part of a broader computational workflow.

Key Differences: Table 9.7 confirms that QPPTRNG's approach offers greater flexibility and integration potential compared to Nie et al.'s (2015) analog photonic system. The QPPTRNG's rigorous validation and compatibility with Qiskit make it more accessible for research and development, while Nie et al.'s high-speed output suits specialized high-throughput applications.

9.4.5 Summary of Comparison

The QPPTRNG demonstrates performance comparable to Kumar et al.'s 2022 QRNG, with both achieving near-maximal min-entropy and passing rigorous statistical tests. The QPPTRNG's phase modulation, however, introduces additional state complexity, potentially enhancing its suitability for phase-sensitive cryptographic applications. Compared to Ozaner et al.'s 2022 implementation, the QPPTRNG offers a more advanced design and comprehensive validation, making it a robust candidate for high-security applications. In contrast to Tamura and Shikano, 2022, the QPPTRNG's gate-based design ensures accessibility and

Table 9.7 Comparison of QPPTRNG with Related QRNG Studies

Aspect	*QPPTRNG (Proposed Work)*	*Kumar et al. (2022)*	*Ozaner et al. (2022)*
Platform	IBM Qiskit	IBM Qiskit	IBM Qiskit
Quantum Gates	RY(π/2) + S Gate	Hadamard Gate	Hadamard Gate
Qubit Count	Scalable (n qubits)	24 qubits	Not specified
Statistical Tests	NIST SP 800-22, 800-90B, Dieharder, ENT	NIST SP 800-22, 800-90B	Not Reported
Min-Entropy	0.9983 (Markov), 0.9985 (multi-Markov)	0.999507 (worst-case)	Not Reported
Additional Tests	Restart, autocorrelation	Restart, autocorrelation	None
Advantages	Phase modulation, scalability, efficiency	High entropy, simple design	Basic quantum randomness
Limitations	Two Gates per qubit	Higher gate count for large systems	Limited validation, basic design

scalability. At the same time, Nie et al.'s 2015 photonic QRNG excels in speed but lacks integration with quantum computing platforms. These comparisons highlight the QPPTRNG's unique contribution to QRNG research, combining efficiency, scalability, and phase-enhanced randomness for post-quantum cryptographic protocols.

9.4.6 Limitations and Challenges

The QPPTRNG, despite its strengths, faces challenges inherent to current quantum hardware. Superconducting qubits suffer from decoherence, with T1 and T2 times of 50–200 femtoseconds (fs), which limits circuit depth. Gate errors (0.1–1%) and readout errors (5%) can introduce noise, potentially skewing randomness quality. Kumar et al.'s 2022 24-qubit system exacerbates these issues due to the higher qubit count, while Ozaner et al.'s simple design may mitigate errors but lacks validation. Tamura and Shikano 2022 analog approach avoids gate errors but requires precise calibration, while Nie et al., 2015 photonic system faces challenges in miniaturization. Mitigation strategies for the QPPTRNG include Qiskit's error mitigation tools, such as dynamical decoupling and readout error correction, which reduce noise impact by 20–50% in practice. Scalability remains a challenge, as increasing qubit count increases queue times and noise exposure on IBM's cloud-based systems. Dedicated hardware or Qiskit Runtime's optimized execution can address this, reducing latency by up to 100 times. Deployment in real-world systems, such as IoT or satellite communications, requires robust post-processing to maintain randomness quality under noisy conditions, a challenge shared across all QRNGs.

9.5 CONCLUSION AND FUTURE SCOPE

This chapter introduces a QPPTRNG harness that leverages the superposition of quantum gates within a scalable circuit architecture implemented on the IBM quantum platform. The proposed design utilizes RY(π/2) and S gates across multiple qubits, generating highly unpredictable quantum states that serve as a robust source of true randomness. Extensive statistical validation across multiple well-established test suites demonstrates the exceptional quality and cryptographic strength of the random sequences generated. The NIST 800-22 suite confirms the randomness with high p-values in all 13 foundational tests, including a Monobit

Frequency Test p-value of 0.9283 and a Run Test p-value of 0.9777, indicating uniform distribution and temporal independence. Complex tests such as the DFT (p = 0.9415) and Maurer's Universal Statistical Test (p = 0.5332) further support the absence of periodicities and compressible patterns. The NIST 800-90B min-entropy estimations bolster these findings, with near-ideal values from the Markov (0.9983) and Lz78y (0.9982) estimators and strong results in Collision (0.903) and Compression (0.830) tests, underscoring the output's maximal unpredictability and resistance to bias. Complementing this, the Dieharder suite passed all 28 tests, with average p-values ranging from 0.1094 to 0.9863, confirming the absence of linear dependencies, clustering, or correlations. ENT test results reveal perfect entropy (1.0 bits per byte), negligible serial correlation (0.000887), and incompressibility, collectively validating the bitstream's statistical randomness and independence. Together, these results illustrate that the QTRNG's outputs exhibit maximal entropy, minimal bias, and strong temporal and spatial independence, qualities critical for cryptographic security. Unlike classical PRNGs and deterministic algorithmic methods such as cellular automata, this quantum approach leverages fundamental quantum mechanical phenomena, ensuring true non-deterministic randomness that is inherently resistant to prediction or replication.

In summary, the proposed QTRNG establishes a new paradigm for secure, scalable, and verifiable true random number generation, positioning itself as a highly promising candidate for next-generation cryptographic protocols, secure key generation, and quantum-safe communication systems. This work lays the foundation for the practical deployment of quantum-based randomness sources, addressing the growing demand for secure and reliable random numbers in the era of quantum computing.

In the near future, it is planned to extend QPPTRNG by involving fault-tolerant architectures, such as surface codes, to mitigate the impact of noise on advanced quantum systems. Integrating quantum-classical hybrid methods, where QRNG outputs are refined through classical algorithms, can enhance performance in domains such as machine learning and financial simulations. Incorporating entangling gates, such as CNOT or CZ, may enable the generation of correlated random sequences, broadening QPPTRNG's utility in QKD and quantum-secured encryption protocols. Leveraging Qiskit's ongoing developments, including Qiskit Runtime and improved error mitigation techniques, will facilitate these advancements, reinforcing QPPTRNG's role in quantum-safe technologies (Tamura 2020). In summary, QPPTRNG's phase-driven, efficient, and scalable framework, rigorously validated by statistical tests, positions it as a leading QRNG for cryptographic purposes. Its comparative analysis with works by Kumar et al., 2022, Ozaner et al., 2022, Tamura and Shikano 2022, and Nie et al., 2015 highlighted its distinctive combination of accessibility, performance, and randomness quality, setting the stage for further breakthroughs in quantum random number generation.

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Chapter 10

From Observations to Models

Statistical Methods in Cosmology

Simran Arora

10.1 INTRODUCTION

As cosmology has evolved from a field with limited data to one driven by vast information, applying advanced statistical methods has become increasingly important. Statistics is inherently linked to cosmology because theories about the origin of the Universe and evolution do not provide exact predictions for specific events, such as the formation of an individual galaxy at a precise location and time or the actual temperature of a particular region of the cosmic microwave background (CMB). Instead, these theories offer statistical predictions about the overall properties of the Universe, while our observations capture only one possible realization of those predictions (Riess et al., 1998; Perlmutter et al., 1999).

Cosmology is often described as having entered the precision era, where achieving precision necessitates a thorough understanding of error bars and confidence intervals, an inherently statistical concept. However, we should strive for precision and accuracy, a goal cosmology has historically struggled with. Achieving accuracy requires accounting for systematic errors in addition to statistical uncertainties, which also demands the use of advanced statistical techniques. Given these challenges, a solid grasp of fundamental statistical methods has become essential for interpreting recent developments in cosmological research.

The Λ cold dark matter (CDM) model is the most commonly recognized framework for explaining this phenomenon. This theoretical framework systematically integrates a positive cosmological constant Λ into the Einstein–Hilbert action of general relativity. This inclusion helps explain the observed acceleration in the Universe's expansion, especially during its later cosmological stages, which has been validated through numerous astronomical observations, including those of distant supernovae and CMB radiation. This integration enhances our comprehension of cosmic dynamics and encourages deeper exploration into the fundamental nature of dark energy (DE) and its consequences for the Universe's ultimate destiny. Although this model offers a straightforward explanation, it faces significant challenges. Among them is the absence of a theoretical foundation for the observed value of Λ. Consequently, the fundamental physical reason for the accelerated expansion is still unclear. Additionally, the model is affected by what is known as the Hubble tension: a discrepancy between the present value of the Hubble constant H_0, determined through direct, model-independent measurements, and the values derived from cosmological models.

A well-known instance of such an issue is the 5σ discrepancy between the Hubble constant value obtained from direct observations by the SH0ES collaboration (Riess et al., 2022) and the estimate derived from CMB measurements by the Planck collaboration (Aghanim et al., 2020). While this mismatch might suggest that the ΛCDM model is not the correct description of the Universe, it could also be due to systematic errors in the direct measurements. Numerous alternative theories have been proposed to account for the observed accelerated expansion of the Universe. These theories generally fall into three primary categories. The

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first approach involves incorporating an additional fluid into the energy-momentum tensor (EMT), characterized by a time-dependent equation of state (EoS), commonly known as parametric DE models. The second category includes models that introduce a novel field, which may have minimal coupling with either gravity or matter and is also considered within the EMT.

The rise of contemporary cosmological observations and the arrival of diverse data made it essential to assess cosmological models in relation to these observations, a task typically carried out using statistical approaches. However, it is crucial to recognize that since we observe only a single Universe, a frequentist approach to statistics is not ideal; we cannot generate multiple Universes to make statistical inferences in the traditional sense. Instead, Bayesian statistics offers a powerful alternative (Trotta, 2017; Padilla et al., 2021; Leclercq, 2018; Santos et al., 2017).

Probability and statistics are crucial in cosmology for a wide range of applications. For instance, they are employed to examine whether the Universe is homogeneous and isotropic on large scales and to evaluate if the Universe's initial conditions align with a Gaussian distribution. Statistical methods also help detect nonzero tensor modes, which can provide evidence for gravitational waves, and measure cosmological parameters such as the density parameter Ω_m from observational data like Wilkinson Microwave Anisotropy Probe. They play a crucial role in constraining the spectral slope n of the primordial power spectrum at different confidence levels and in determining the optimal value for the DE EoS parameter ω. Moreover, statistical methods are used to compare cosmological models, such as assessing whether a model with an EoS parameter different from $\omega = p_\Lambda / \rho_\Lambda = -1$ fits the data more accurately than one with nonzero curvature. Finally, they are indispensable in forecasting the constraints on parameters like ω for future surveys with specific observational characteristics. These applications demonstrate how vital statistical analysis is for interpreting data and refining cosmological theories (Weinberg, 1989; Sahni and Starobinsky, 2000; Padmanabhan, 2003).

Despite the growing body of observational data, several unresolved discrepancies lacking clear physical explanations remain. This has motivated the development of innovative analytical tools to extract deeper insights from the data while optimizing the use of computational resources. Over the past few decades, machine learning, particularly deep learning, emerged as a powerful approach to data analysis. For example, artificial neural networks (ANNs) had proven to be effective across numerous tasks, including regression, classification, image analysis, and time series forecasting (Goodfellow et al., 2016; Hortua et al., 2020; Spurio Mancini et al., 2022; Gómez-Vargas et al., 2021; de Dios Rojas Olvera et al., 2022).

ANNs have also been used as substitutes for conventional numerical methods for solving physical models. For instance, various studies have utilized ANNs trained via supervised learning on the outcomes of numerical simulations, as illustrated by Spurio Mancini et al. (2022), where they were applied to constrain model parameters using observational data. Alternative methods include ANNs that can directly resolve differential equations, thereby removing the requirement for conventional numerical solvers, as showcased in Piscopo et al. (2019), which builds upon the unsupervised learning method introduced by Lagaris et al. (1998). Unlike conventional numerical methods, ANN-based solutions are smooth and continuously differentiable.

Modeled after biological neural networks, an ANN consists of a system of interconnected neurons arranged in multiple layers (Aggarwal, 2018; Wang et al., 2021; Gómez-Vargas et al., 2023). Each neuron operates based on an activation function that determines its output. These neurons include several hyperparameters and statistical variables that must be appropriately set. This is achieved through training the network with observational data, allowing it to adjust and optimize its responses. As a result, the trained ANN can accurately

replicate observational data across different redshifts. Therefore, in the cosmological context, the training data for the network is created using the redshift distribution of the actual $H(z)$ measurements under the assumption of a spatially flat cosmological model. Additionally, the number of simulated data points matches the number of observed ones. The neural network inputs redshift values and outputs the corresponding Hubble parameter along with its associated uncertainty at each redshift. During training, the network's parameters are refined through a learning process that leverages the available observational data sets.

Objectives of the Chapter

The objectives of the chapter are:

- **Hypothesis Testing:** The first set of questions addresses hypothesis testing, where the objective is to assess whether the data support a given hypothesis. In cases such as detection problems, this often involves testing the null hypothesis, assuming a specific value (such as zero), and evaluating whether the data are consistent with that assumption.
- **Parameter Estimation:** The next set of questions focuses on parameter estimation, where a model is defined with certain free parameters, and the goal is to measure these parameters based on available data.
- **Model Testing:** A further important aspect is model evaluation, which entails assessing various models to identify which one offers a more accurate interpretation of the data. This comparison can vary depending on whether the models have different numbers of parameters, share common features, or are based on fundamentally different approaches.
- **Forecasting:** Ultimately, forecasting is essential for anticipating the possible results of upcoming experiments and enhancing experimental design by evaluating the expected performance beforehand.

This chapter focuses on analysis techniques designed to handle the complexity of modern cosmological data, which can also be applied to many other fields involving large datasets. Using a straightforward example, it presents the basic ideas of likelihood, prior, and posterior. The section subsequently summarizes one of the prevalent uses of these ideas in cosmology, particularly in how they help restrict cosmological parameters.

Key methods for addressing challenges in cosmology include the utilization of the Fisher matrix and Markov Chain Monte Carlo (MCMC) sampling. The Fisher matrix provides a quick approach to estimating parameter error bars, giving insight into the level of precision that a specific experimental configuration can achieve. On the other hand, MCMC sampling allows for the effective identification of optimal parameters and the assessment of related uncertainties, even in the presence of intricate and non-Gaussian likelihood functions. These statistical techniques extend beyond cosmology and astrophysics and have wide-ranging applications in various scientific fields where parameter estimation and model assessment are crucial.

Organization of the chapter

The rest of the chapter is organized as: Section 10.2 provides an overview of Bayesian statistics. Section 10.3 focuses on MCMC techniques used for parameter inference, while Section 10.4 explains the core principles of cosmology. Section 10.5 explores two cosmological

models within the framework of a flat Friedmann–Lemaître–Robertson–Walker (FLRW) metric. Furthermore, Section 10.6 introduces the background of machine learning. Section 10.7 describes the implementation of ANN. And, finally, section 10.8 concludes the chapter.

10.2 BAYESIAN STATISTICS

Prior to examining the application of Bayesian statistics in cosmology, it is beneficial to first understand the fundamental mathematical concepts that form the basis of the Bayesian approach.

10.2.1 Bayes' Theorem

Bayes' theorem is derived directly from the basic axioms of probability. Let x represent a random variable that identifies a particular event of interest within a certain probability space. The probability distribution linked to this random variable is denoted as $P(x)$, which is systematically characterized as follows:

$$P(x) \geq 0, \tag{10.1}$$

$$\int_{-\infty}^{\infty} xP(x) = 1, \tag{10.2}$$

$$P\left(x_1 \bigcup x_2\right) = P(x_1) + P(x_2). \tag{10.3}$$

Typically, when one event occurs assuming that another has previously occurred, the combined probability of x_1 and x_2 can be expressed as the multiplication of the probability of x_1 and the conditional probability of x_2 given x_1:

$$P\left(x_1 \bigcap x_2\right) = P(x_1)P(x_2 \,/\, x_1). \tag{10.4}$$

Within the Bayesian perspective, the data and the model are treated as elements of a unified framework. For a specific model M, let $\mathcal{D}$ represent the data that has been observed, and θ signify the vector of parameters for the model. The expression can then be reformulated as:

$$P(\theta \mid \mathcal{D}, M) = \frac{P(\mathcal{D} \mid \theta, M)P(\theta \mid M)}{P(\mathcal{D} \mid M)}$$

The aforementioned equation is called Bayes' theorem and is a crucial element of Bayesian inference. In this framework, $P(\theta \mid \mathcal{D}, M)$ denotes the posterior probability of the parameters of the model. The likelihood, represented as $L(\mathcal{D} \mid \theta, M) \equiv P(\mathcal{D} \mid \theta, M)$, will be the main focus of the subsequent section. The term $\Pi(\theta) \equiv P(\theta \mid M)$ is referred to as the prior, encapsulating our initial understanding or assumptions about the model parameters before any data is taken into account. This prior can be informed by previous experimental findings or

theoretical considerations. The quantity $P(\mathcal{D} \mid M)$ is called the evidence, or Bayesian evidence, which plays a crucial role in model comparison.

The prior reflects the knowledge about the model parameters before observing any data. It can be expressed in different manners, one popular option being the uniform prior, which is also known as the flat prior.

$$\Pi(\theta) \propto c. \tag{10.5}$$

In this case, c is a constant. This prior suggests that all values of the parameters are assumed to be equally probable before any data is observed. Choosing this prior suggests that the posterior probability will be entirely dictated by the likelihood function.

Rewriting Bayes' theorem to explicitly incorporate prior information F, the following expression is obtained:

$$P(\theta / \mathcal{D}F, M) = \frac{P(\theta / F, M)P(\mathcal{D}F / \theta, M)}{P(\mathcal{D} / F, M)}, \tag{10.6}$$

Next, by introducing a new dataset $\mathcal{D}'$ and incorporating the previous data into the prior information $F' = \mathcal{D}F$, the following expression is obtained:

$$P(\theta / \mathcal{D}'F', M) = \frac{P(\theta / F, M)P(\mathcal{D}\mathcal{D}'F / \theta, M)}{P(\mathcal{D}\mathcal{D}' / F, M)}, \tag{10.7}$$

which shows the equivalence of the two approaches.

Now, let us move on to another important tool: the likelihood.

10.2.2 The Likelihood

In Bayesian inference, the Bayesian evidence is often disregarded when analyzing the parameter space of a single model. Therefore, we can set it to $P(\mathcal{D} / M) = 1$ without any loss of generality. If we neglect the prior approach that will be further justified in the next section, the posterior distribution becomes proportional to the likelihood; that is, $P(\theta / \mathcal{D}, M) \propto L(\mathcal{D} / \theta, M)$. As a result, maximizing the likelihood enables the identification of the most likely parameter values for the model based on the observed data.

Assuming a distribution with a single peak, $\tilde{\theta}$ is defined as the mean of the posterior distribution.

$$\tilde{\theta} = \int d\theta\theta P(\theta / \mathcal{D}, M).$$

If the model under consideration is accurately specified and the expected value of $\tilde{\theta}$ aligns with the true or most likely parameter value θ_0, then $\langle \tilde{\theta} \rangle = \theta_0$. In this case, $\tilde{\theta}$ is considered an unbiased estimator.

To approximate the likelihood near its peak, $ln\,\mathcal{L}(\mathcal{D} / \theta)$ is expanded in a Taylor series around θ_0.

$$\ln \mathcal{L}(\mathcal{D} \mid \theta) = \ln \mathcal{L}(\mathcal{D} \mid \theta_0) + \frac{1}{2}(\theta_i - \theta_{0i}) \frac{\partial^2 \ln \mathcal{L}}{\partial \theta_i \partial \theta_j} (\theta_j - \theta_{0j}) + \ldots,$$

represents the true parameter values of the model.

This estimation shows that the probability can be represented as a multivariate Gaussian distribution.

$$\mathcal{L}(\mathcal{D} \mid \theta) = \mathcal{L}(\mathcal{D} \mid \theta_0) \exp\left(-\frac{1}{2}(\theta_i - \theta_{0i}) H_{ij} \left(\theta_j - \theta_{0j}\right)\right), \tag{10.10}$$

where $H_{ij} = -\frac{\partial^2 \ln \mathcal{L}}{\partial \theta_i \partial \theta_j}$ is the Hessian matrix of the log-likelihood, describing the curvature of the likelihood function around θ_0. It determines whether there is a correlation between θ_i and θ_j. If the matrix is diagonal, the estimates are independent of each other.

The given likelihood expression acts as a useful approximation when the posterior distribution is unimodal (exhibiting a single peak). If the data errors follow a normal distribution, the likelihood function will also take a Gaussian form. This is especially true when the model has a linear dependence on its parameters. The central limit theorem remains applicable even if the data does not follow a normal distribution. This theorem states that, given specific conditions, the resulting probability distribution can be effectively approximated by a multivariate Gaussian distribution, defined by its mean vector and covariance matrix. This approximation is particularly valid in high-dimensional environments, where the central limit theorem aids in converging to a Gaussian shape, thus enabling more precise modeling and analysis of intricate data sets.

An essential concept is now introduced for the statistical comparison of models.

10.2.3 Chi-Square and Goodness of Fit

Think about a set of observations and a model defined by a group of parameters θ, where the objective is to align the model with the data. The model might be derived from physical laws or chosen for its simplicity and practicality. To evaluate the model's fit to the data, a merit function should be introduced, which quantifies the level of agreement. Maximizing this function provides the optimal parameter values.

A robust fitting procedure should deliver:

- The optimal parameter values,
- An estimation of the uncertainties associated with these parameters, and
- Ideally, a quantification of the goodness of fit.

It is crucial to recognize that, if the model does not adequately capture the data, the resulting best-fit parameters may lack meaningful interpretation or relevance.

Utilizing the Gaussian approximation outlined in Equation (10.10), it is observed that the likelihood reaches its peak when the value of

$$\chi^2 \equiv (\theta_i - \theta_{0i}) H_{ij} \left(\theta_j - \theta_{0j}\right), \tag{10.11}$$

is minimized. The quantity χ^2, known as chi-square, is connected to the Gaussian likelihood as follows:

$$\mathcal{L} = \mathcal{L}_0 \, e^{-\chi^2/2}. \tag{10.12}$$

It follows that optimizing the Gaussian likelihood is equivalent to minimizing the chi-square statistic. However, as previously noted, there are cases in which a Gaussian distribution does not adequately represent the likelihood. In such situations, the chi-square and likelihood functions are not interchangeable and must be treated distinctly.

When examining the distribution of χ^2 values in relation to its minimum, it adheres to a chi-square distribution characterized by $\nu = n - m$ degrees of freedom, where n denotes the number of independent data points and m is the count of parameters in the model. This distribution allows us to determine the likelihood that a given χ^2 value surpasses a certain threshold $\tilde{\chi}^2$ due to random variability, assuming the model holds true. The probability is expressed as:

$$\mathcal{F}\left(\nu, \tilde{\chi}^2\right) = 1 - \Gamma\left(\frac{\nu}{2}, \frac{\tilde{\chi}^2}{2}\right). \tag{10.13}$$

In this context, Γ denotes the incomplete Gamma function. The likelihood that the observed χ^2 value is less than $\tilde{\chi}^2$ can be expressed as $1 - \mathcal{F}$. This relationship is valid when the errors are normally distributed, and the model is linear in terms of the likelihood, as is the case with Gaussian likelihoods.

By evaluating $\mathcal{F}$ at the best-fit values (where χ^2 is minimized), the goodness of fit can be assessed. A small value of $\mathcal{F}$ (indicating low probability) suggests the following: the model may not be correct and should be reconsidered; the error estimates could be too small; and the error measurements might not be normally distributed.

Alternatively, a large $\mathcal{F}$-value could suggest that:

- The errors have been overestimated.
- The data might be correlated or not independent.
- The data distribution may not follow a Gaussian form.

10.2.4 Confidence Regions

After determining the best-fit parameters, the next step is to identify the confidence regions where the parameter values are likely to be good candidates for the model. A reasonable approach is to select values within a small area around the best-fit point. One common method is to define these regions based on constant χ^2 values. When the χ^2 function has multiple minima, the resulting confidence regions are considered non-connected. For multivariate Gaussian distributions, like the likelihood approximation, the corresponding regions are typically ellipsoidal. This section outlines the method for calculating these confidence regions.

A slight departure from the optimal chi-square value is evaluated, represented as $\Delta\chi^2 = \chi^2 - \chi^2_{\text{min}}$. Using the properties of the χ^2 distribution, confidence regions can be defined based on changes in χ^2 near its minimum. In the case of Gaussian distributions, these areas relate to the standard confidence intervals of 1, 2, and 3σ, which represent 68.3%, 95.4%, and 99.73%, respectively.

Follow a systematic procedure to calculate constant χ^2 confidence regions. First, identify the best-fit parameters by minimizing the χ^2 value or by maximizing the likelihood function. Once you have these parameters, it's important to check that the Q-value associated with the best-fit solution meets satisfactory criteria. After this verification, you can proceed with the next steps.

- Let m denote the count of parameters, n signify the number of data points, and p indicate the required confidence level.

- Find the solution to the equation.

$$Q\left(n-m, \min\left(\chi^2\right)+\Delta\chi^2\right)=p.$$

- Identify the parameter region where $\chi^2 \leq \min\left(\chi^2\right)+\Delta\chi^2$. This defines the confidence region.

10.2.5 Fisher Matrix

It is essential to determine the precision with which the parameters can be estimated. Fisher proposed a method to address this more than 70 years ago. To start, let us revisit the Gaussian likelihood. The Hessian matrix H_{ij} details the covariance and uncertainties related to the parameters. When all parameters are kept constant except for one (for example, the j-th parameter), the uncertainty associated with that parameter can be represented as $1/\sqrt{H_{jj}}$. These uncertainties are known as conditional errors, although they are not commonly used.

A way to assess the accuracy of a model, which arises naturally within the framework of Gaussian likelihoods, is the Fisher information matrix, which is defined as.

$$F_{ij}=-\left\langle\frac{\partial^2\mathcal{L}}{\partial\theta_i\partial\theta_j}\right\rangle,\ L=\ln\mathcal{L}. \tag{10.14}$$

It is evident that F denotes the average of the Hessian matrix H, with the average calculated based on the observational data.

When the datasets are not related to each other, the overall likelihood is calculated by multiplying the individual likelihoods, whereas the Fisher matrix is obtained by adding together the Fisher matrices from each dataset. A straightforward scenario to analyze is when only a single parameter θ_i and the likelihood adhere to a Gaussian distribution. The variation in the log-likelihood is expressed as:

$$\Delta\mathcal{L}=\frac{1}{2}F_{ii}\left(\theta_i-\theta_{0i}\right)^2, \tag{10.15}$$

and for $2\Delta\mathcal{L}=1$, which corresponds to the 68% confidence interval, we find that $1/\sqrt{F_{ii}}$ represents the 1σ error for θ_i. In a more general setting, the covariance between parameters is:

$$\sigma_{ij}^2\geq\left(F^{-1}\right)_{ij}, \tag{10.16}$$

This implies that when parameters are estimated simultaneously from the data, the marginalized uncertainty for θ_i is:

$$\sigma_{\theta_i}\geq\sqrt{\left(F^{-1}\right)_{ii}}. \tag{10.17}$$

The benefit of utilizing the Fisher matrix approach is its ease of use. Knowledge of the model and measurement uncertainties provides a straightforward approach to constructing the Fisher matrix. Assuming a Gaussian likelihood, the Fisher matrix acts as the inverse of the covariance matrix, revealing critical insights into parameter estimation. By meticulously calculating the Fisher matrix, we derive the covariance matrix, which in turn unveils the uncertainties

associated with our model parameters, providing essential information for assessing the reliability of our results. Additionally, this approach is computationally efficient, enabling the evaluation of different experimental configurations and the optimization of experimental design.

Note. This chapter primarily focuses on Gaussian likelihoods. It is crucial to recognize that if the data errors exhibit a Gaussian distribution, the likelihood function for the data will naturally assume the structure of a multivariate Gaussian. When the data does not follow a Gaussian distribution but comes from a distribution with a finite variance, the central limit theorem can still be utilized. By grouping the data into bins, with each bin consisting of multiple independent measurements, the central limit theorem guarantees that the error distribution for every bin will resemble a multivariate Gaussian distribution.

However, even when the data are Gaussian, it doesn't necessarily mean that the likelihood surface for the parameters will also be multivariate Gaussian. For this to consistently be the case, the model must exhibit a linear relationship with the parameters. Remarkably, the Gaussian approximation often still applies, even without depending on the central limit theorem, and can be valid even when beginning with a distribution that is significantly non-Gaussian.

10.3 MCMC TECHNIQUES FOR PARAMETER INFERENCE

Numerical techniques are essential for parameter estimation, and while there are several approaches available, this section focuses on the MCMC method with the Metropolis–Hastings algorithm (MHA). Let us consider some important tips that can help improve the efficiency of our calculations. The main goal of the MCMC algorithm is to generate an extensive sequence of points, known as a chain, that efficiently navigates the parameter space. This technique facilitates the precise estimation of the posterior distribution, as described in Equation (10.7). By methodically sampling from the parameter space, the algorithm guarantees that the produced points are representative of the underlying distribution, which supports strong statistical inference. In this section, we provide a basic overview of the process, though for a more detailed and formal explanation of MCMC theory, we recommend consulting references (Esquivel et al., 2021; Ross, 2014).

Monte Carlo methods utilize random number generation to estimate a specific quantity. A sequence $X_1, X_2, \ldots$ is classified as a Markov Chain if the conditional probability distribution of X_{n+1}, given all previous values $X_1, X_2, \ldots, X_n$, depends only on the current value X_n. In simpler terms, a Markov Chain is a stochastic process in which the subsequent state is solely determined by the current state, with no effect from prior steps.

A key feature of a Markov Chain is its tendency to eventually reach a stationary distribution, where the successive states serve as representative samples from the target distribution, specifically the posterior $P(\theta \mid \mathcal{D}, M)$. This allows the generated chain to be used for estimating quantities like the mean, variance, and other statistical measures of interest. By integrating these methods, we obtain what is known as MCMC. Typically, the quantity of samples needed to achieve dependable estimates via MCMC is approximately related to the dimensions of the parameter space. This characteristic makes MCMC especially useful for addressing challenges that involve numerous parameters.

As outlined earlier, in a Markov Chain, each new point θ_{i+1} is proposed based on the current point θ_i. An essential part of this process involves creating a guideline to determine if the suggested point should be accepted or turned down, based on whether it provides a more accurate portrayal of the model. If the new point results in a worse model, it may still be accepted. This is important because only accepting steps that improve the model could lead to

convergence at a local maximum in the parameter space, thus restricting the exploration to a limited region. The simplest algorithm that incorporates this concept is known as the MHA.

In the MHA (Hastings, 1970), the procedure starts with a randomly selected initial state θ_i, which corresponds to an initial posterior probability $p_i = p(\theta_i \mid \mathcal{D}, M)$. A new candidate state θ_k is then generated using a proposal distribution $q(\theta_i, \theta_k)$, responsible for producing random steps. The likelihood of agreeing to this suggested point is determined by:

$$p(\text{acceptance}) = \min\left(1, \frac{p_k q(\theta_k, \theta_i)}{p_i q(\theta_i, \theta_k)}\right), \tag{10.18}$$

where p_k is the posterior probability at the candidate point θ_k and p_i is the posterior probability at the current point θ_i.

When the proposal distribution is symmetric, the algorithm simplifies to the standard Metropolis method, whereby the acceptance probability can be represented as:

$$p(\text{acceptance}) = \min\left(1, \frac{p_k}{p_i}\right). \tag{10.19}$$

Alongside the MHA, various other sampling methods exist for producing elements within a Markov Chain. The generation of these elements is inherently probabilistic, depending on the algorithm being used. Although the MHA is among the most straightforward methods utilized in Bayesian analysis, numerous alternative algorithms are available that can also produce the intended outcomes. Some of the most well-known and efficient alternatives include the EMCEE, Gibbs sampling, Montepython, and CosmoMC (Foreman-Mackey et al., 2013; Brinckmann and Lesgourgues, 2019; Lewis and Bridle, 2002).

10.4 COSMOLOGICAL MODEL

This section presents the fundamental principles of cosmology and demonstrates how Bayesian statistics can be applied for parameter estimation (Verde, 2010). The illustrations provided will mainly pertain to the homogeneous background Universe, deliberately omitting perturbations, as the main aim is to showcase the application of these statistical methods rather than delve into complex cosmological phenomena.

A fundamental assumption that characterizes the Universe's geometry is the cosmological principle. This principle states that, at any specific moment and over sufficiently large distances, the observable Universe can be effectively modeled as homogeneous and isotropic with a high degree of accuracy (Amendola and Tsujikawa, 2010).

Assuming isotropy and the idea that our position in the Universe is not unique, a concept referred to as the Copernican Principle, homogeneity naturally follows. Isotropy at each point implies a uniform distribution of matter and energy. Homogeneity suggests that the Universe appears identical at every location in space, while isotropy indicates that it looks the same in every direction.

10.4.1 Einstein Field Equations

The framework of general relativity defines the connection between the shape of spacetime and the arrangement of matter contained within it. In essence, the curvature governs the

motion of matter, leading to gravitational effects, while matter and energy dictate the curvature. This connection is mathematically represented by the *EMT* $T_{\mu\nu}$, which describes how energy and momentum are distributed and move throughout spacetime.

This idea can be elegantly summarized as follows:

> Matter influences the curvature of spacetime, while spacetime directs the movement of matter.

Mathematically, this is captured by the Einstein field equations:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{10.20}$$

where $G_{\mu\nu}$ represents the Einstein tensor, encapsulating spacetime curvature, and $8\pi G$ is the proportionality constant relating geometry to energy-momentum. In this detailed analysis, natural units are employed to simplify the calculations, where the speed of light c and the reduced Planck constant $\hbar$ are equal to unity. This means that $c = 1$ and $\hbar = 1$, facilitating a more straightforward representation of physical equations and concepts in theoretical physics.

This convention simplifies equations by eliminating unnecessary constants, making the fundamental physical relations more transparent.

The separation between two points in curved spacetime is described by the line element:

$$ds^2 = g_{ij} dx^i dx^j, \tag{10.21}$$

where g_{ij} denotes the metric tensor, which encapsulates all the geometric information about spacetime.

From this point onward, unless otherwise specified, the following conventions are adopted:

- Greek indices $(i, j, \ldots)$ range from 0 to 3 and represent spacetime coordinates.
- Latin indices $(i, j, \ldots)$ range from 1 to 3 and correspond to spatial coordinates.

The geometry that most accurately characterizes a homogeneous, isotropic, and expanding Universe is represented by the FLRW metric. The associated line element is

$$ds^2 = dt^2 - a^2(t) K_{ij} dx^i dx^j, \tag{10.22}$$

where

$$K_{ij} \equiv \delta_{ij} + \kappa \frac{x^i x^j}{1 - \kappa\left(x^k x_k\right)}. \tag{10.23}$$

The scale factor of the Universe, denoted as $a(t)$, is a time-dependent function that is set to one at the present time, meaning $a(t_0) \equiv 1$. Additionally, x^i is the spatial (or comoving) coordinate, the Kronecker delta is represented by δ_{ij}, while κ represents the curvature of spacetime.

10.4.2 Friedmann Equations

The matter composition of the Universe must adhere to the principles of homogeneity and isotropy. Consequently, it is represented by the EMT of an ideal fluid.

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu - P g_{\mu\nu}. \tag{10.24}$$

In this context, ρ represents the energy density, P refers to the pressure of the fluid, and u_μ denotes the four-velocity as it relates to the observer.

Assuming a comoving observer, u_μ simplifies to

$$u^\mu = (1,0,0,0). \tag{10.25}$$

Hence, the EMT simplifies to

$$T^\mu_\nu = g^{\mu\gamma}T_{\gamma\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -P \end{pmatrix}. \tag{10.26}$$

Using Equations (10.4.2) and (10.26), assuming the FLRW metric, the Friedmann equations can be obtained as:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i - \frac{\kappa}{a^2}, \tag{10.27}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\sum_i (\rho_i + 3P_i). \tag{10.28}$$

In these formulas, H symbolizes the speed at which the Universe is expanding or contracting, which is often referred to as the Hubble parameter.

By merging Equations (10.27) and (10.28), the continuity equation can be derived as

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0. \tag{10.29}$$

Equation (10.29) expresses the conservation of the EMT, represented as $\nabla_\mu T^{\mu\nu} = 0$. To complete the system, it is necessary to introduce an EoS that establishes the connection between pressure and energy density for a particular fluid. Consequently, attention is focused on barotropic fluids, which are commonly described by the equation $P = \omega\rho$.

10.4.3 Dynamics of the Universe

The standard ΛCDM model is one of the most widely endorsed frameworks for understanding the Universe. The key components of this model are as follows:

- **Dust:** This component shows zero pressure, which gives the relation for the energy density as $\rho \propto a^{-3}$. Dust consists mainly of baryonic matter, a type of matter we encounter in everyday life.

- **Dark Matter:** It is proposed to account for various observations in astrophysics, such as the movements of galaxies in the Coma cluster and the rotation profiles of galaxies. In the ΛCDM framework, dark matter is thought to interact only through gravitational forces (and potentially weak interactions) with the rest of the Universe, which is why it is referred to as CDM. Potential candidates for Dark Matter comprise weakly interacting massive particles, gravitationally interacting massive particles, axions (hypothetical fundamental particles), and sterile neutrinos.
- **Radiation:** These pertain to relativistic particles that adhere to the equation $P = \frac{1}{3}\rho$, demonstrating a density behavior of $\rho \propto a^{-4}$.
- **Dark Energy:** DE is accountable for the ongoing rapid expansion of the Universe. In the ΛCDM model, which is the dominant cosmological framework for understanding the Universe, DE is denoted by the cosmological constant Λ. This constant is essential for accounting for the accelerated expansion observed in the Universe. Furthermore, DE can also be defined using an EoS $\omega = -1$, where ω describes the connection between pressure and density, signifying that DE functions like a vacuum energy with a uniform density throughout space.

Every component can be defined by its state equation along with the associated density parameter.

$$\Omega_i \equiv \frac{\rho_i}{\rho_{\text{crit}}},$$

with $\rho_{\text{crit}} = \frac{3H^2}{8\pi G}$ being the critical density for a flat Universe (or zero curvature), Equation (10.27) can be rewritten as

$$\frac{H^2}{H_0^2} = \Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{k,0}a^{-2} + \Omega_{\Lambda,0}, \tag{10.30}$$

Here $\Omega_{r,0}$ represents the radiation density parameter, while $\Omega_{m,0} \equiv \Omega_{b,0} + \Omega_{\text{DM},0}$ denotes the total matter density parameter, where $\Omega_{b,0}$ corresponds to baryons and $\Omega_{\text{DM},0}$ to dark matter. The parameter for curvature density is defined as $\Omega_k \equiv -\frac{\kappa}{aH^2}$, while $\Omega_\Lambda \equiv \frac{\Lambda}{3H^2}$ signifies the impact of the cosmological constant. The subscript zero signifies that these values are taken at the current time.

10.4.4 Observational Datasets

This section presents an informative overview of several recent datasets that have been released, including CCs, supernovae, and BAOs. These datasets play a crucial role in enhancing our understanding of the Universe and its expansion.

10.4.4.1 Cosmic Chronometers

These galaxies represent the most substantial masses at varying redshifts z and contain the oldest stellar populations (Yu et al., 2018). They are often utilized for estimating the Hubble factor, as they have minimal amounts of stardust, which aids in acquiring their spectra. The approach involves selecting two galaxies at different redshifts ranging from $z \sim 0$ to $z \sim 2$

and analyzing the upper limit in their age distributions. By doing this, one can calculate the age difference Δt and the redshift difference Δz, allowing for the approximation of the expression $\frac{dz}{dt}$,

$$\frac{dz}{dt} \approx \frac{\Delta z}{\Delta t}.$$

This pertains to the Hubble factor by

$$H(z) = -\frac{1}{1+z}\frac{dz}{dt} \approx -\frac{1}{1+z}\frac{\Delta z}{\Delta t}. \tag{10.31}$$

10.4.4.2 Supernovae Type Ia (SNeIa)

This kind of supernova happens in binary star systems, where one of the stars is a white dwarf that gathers material from its partner. Once the white dwarf reaches a critical mass (approximately 1.4 solar masses), the core temperature becomes high enough to initiate carbon fusion, releasing tremendous energy that triggers the supernova. Type Ia supernovae are thought to happen near a mass threshold of 1.4 solar masses, referred to as the Chandrasekhar mass, leading to their peak brightness being relatively consistent. Their uniformity enables them to serve as standard candles. Research on Type Ia supernovae conducted by the Supernova Cosmology Project and the High-z Supernova Search Team has shown that the Universe is expanding at an accelerating pace (Brout et al., 2022).

In this analysis, chi-square values are calculated using the formula,

$$\chi^2_{SN} = \Delta\mu^T \left(C^{-1}_{stat+sys}\right)\Delta\mu. \tag{10.32}$$

In this context, $C^{-1}_{\text{stat+sys}}$ signifies the inverse of the covariance matrix for the supernova dataset, which includes both statistical and systematic uncertainties. The distance residual, represented as $\Delta\mu$, is defined by $\Delta\mu_i = \mu_i - \mu_{\text{th}}(z_i)$, where μ_i refers to the observed distance modulus and $\mu_{\text{th}}(z_i)$ indicates the theoretical distance modulus at redshift z_i.

It should also be emphasized that $\mu_i - m_i - M$, with m_i denoting the apparent magnitude and M signifying the fiducial magnitude of the ith Type Ia supernova. The theoretical definitions of distance moduli are as follows:

$$\mu^{th} = 5\log_{10} d_L(z) + \mu_0, \mu_0 = 5\log_{10}\frac{H_0^{-1}}{\text{Mpc}} + 25, \tag{10.33}$$

where

$$d_L(z) = \frac{c(1+z)}{H_0}\left(H_0\int_0^z \frac{d\bar{z}}{H(\bar{z})}\right).$$

The study employs the Pantheon+ (Scolnic et al., 2022) dataset, which is among several prominent datasets on supernovae accessible to researchers. Other important datasets consist

of Union 2 (Li et al., 2011), Pantheon (Scolnic et al., 2018), among others. Each of these datasets offers significant insights into the phenomena related to supernovae.

10.4.4.3 Baryon Acoustic Oscillations

This indicates a statistical characteristic noted in the correlation function or power spectrum of galaxies (Blake et al., 2011). The early Universe is modeled as a plasma consisting of coupled photons and matter, including baryons and dark matter. The interplay between gravity and radiation pressure resulted in spherical waves within the plasma.

As the Universe decreased in temperature, protons and electrons came together to create hydrogen atoms, enabling photons to separate from baryonic matter. This decoupling enabled photons to travel freely, while gravity caused matter to accumulate at the centers of the spherical waves. This process led to an accumulation of matter at the center and a shell of baryons at a constant radius, known as the sound horizon. The sound horizon, denoted as r_s, can be detected at the epoch of photon decoupling with a redshift z_*, and acts as a standard ruler while establishing the scale for BAO.

$$r_s = \frac{c}{\sqrt{3}} \int_0^{\frac{1}{1+z_*}} \frac{da}{a^2 H \sqrt{1+\left(\frac{3\Omega_{b0}}{4\Omega_{\gamma 0}}\right)a}}, \tag{10.35}$$

where Ω_{b0} represents the current baryon mass density, and $\Omega_{\gamma 0}$ signifies the present photon mass density. The redshift corresponding to photon decoupling is represented by z_*, which is defined as $z_* = 1091$. To derive the BAO constraints, we utilize the "acoustic scale" $l_A = \pi \frac{d_A(z_*)}{r_s(z_*)}$, where $d_A(z)$ denotes the angular diameter distance in comoving coordinates:

$$d_A(z) = \int_0^z \frac{dz'}{H(z')}. \tag{10.36}$$

The scale of dilation $D_V(z)$ is as follows:

$$D_V(z) = \left[\frac{d_A(z)^2 cz}{H(z)}\right]^{1/3}. \tag{10.37}$$

Ultimately, the BAO constraints are derived using the following ratio:

$$\frac{d_A(z_*)}{D_V(z_{BAO})}. \tag{10.38}$$

A vast amount of BAO data exists, featuring surveys from SDSS, BOSS, and DESI (for full details on these data, refer to Calderon et al., 2024; Hou et al., 2021). This analysis focuses

on data from DESI 2024. By utilizing various combinations of these datasets, cosmological models can be refined and constrained with greater precision.

10.5 COSMOLOGICAL MODELS

The ΛCDM model has proven to be highly effective in explaining a diverse range of observations in cosmology and astrophysics. Its defining parameters, often referred to as standard cosmological parameters, serve as fundamental descriptors of the Universe's large-scale properties. While these parameters are not derived from first principles, their values are obtained by calibrating the model against observational datasets to achieve the best fit. When constructing a model of a Universe that exhibits both homogeneity and isotropy, we rely on the density parameters of various components, denoted as $\Omega_{i,0}$, together with the present-day Hubble constant H_0, which are linked through Equation (10.30). Among these, the radiation density parameter Ω_γ is determined with exceptional precision from measurements, rendering it unnecessary to treat this parameter as a free variable in model fitting.

This study investigates models from both categories within the framework of a flat FLRW metric. A brief overview of the scenarios considered in the analysis is provided below.

10.5.1 Model A

General relativistic cosmological models that incorporate a constant EoS parameter for DE, denoted by ω, extend the conventional ΛCDM paradigm, in which $\omega = -1$. Within this cosmological framework, the dimensionless Hubble parameter takes the form:

$$E(z)^2 = \Omega_{m,0}a^{-3} + \Omega_{\Lambda,0}a^{-3(1+\omega)}. \tag{10.39}$$

The expression $E(z) = \dfrac{H(z)}{H_0}$ represents the Hubble parameter in a dimensionless form. The symbols $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ indicate the present-day values of the density parameters for matter, which includes both baryonic and dark components, and for DE, respectively. These parameters satisfy the condition for flatness:

$$\Omega_{\Lambda,0} = 1 - \Omega_{m,0}, \tag{10.40}$$

where H_0 represents the Hubble constant, $\Omega_{m,0}$ denotes the matter density parameter at the current time, and $\Omega_{\Lambda,0}$ corresponds to the cosmological constant contribution, with the condition that $\Omega_{m,0} + \Omega_{\Lambda,0} = 1$ in a flat Universe. To fit the model to observational data, the relation $a = \frac{1}{1+z}$ is employed, where z is the redshift. Since all observational data are obtained in terms of redshifts, this transformation allows us to express the cosmological equations in terms of z, making them directly comparable to observational constraints.

10.5.2 Model B

A broader category of models arises when the EoS for DE is permitted to change depending on the scale factor a. With this assumption, the Friedmann equation is simplified to:

$$E(z)^2 = \Omega_{m,0}a^{-3} + \Omega_{\Lambda,0}\exp\left[3\int_1^a \frac{1+\omega(a')}{a'}da'\right], \tag{10.41}$$

where $\omega(a')$ is the scale factor-dependent EoS parameter for DE.

To distinguish a time-varying DE component from Λ, we examine two specific forms of the EoS $\omega(a)$. One frequently used model is the Chevallier–Polarski–Linder parametrization (Linder, 2003), which can be written as:

$$\omega(a) = \omega_0 + \omega_a(1-a). \tag{10.42}$$

Here ω_0 denotes the current value of the EoS parameter, while ω_a characterizes the rate at which it changes with time.

Substituting this form into Equation (10.41), the DE contribution simplifies to:

$$\Omega_{\Lambda,0} a^{-3(1+\omega_0+\omega_a)} \exp\left[-3\omega_a(1-a)\right]. \tag{10.43}$$

For these specific examples, only the CC, DESI (BAO), and SNeIa (Pantheon+) are considered, as illustrated in Figure 10.3. The model is constrained using the combined dataset comprising BAO and SNeIa measurements.

The parameters H_0 and $\Omega_{m,0}$ are subsequently estimated. Assuming a Gaussian distribution, the likelihood function can be formulated in the following manner:

$$\mathcal{L} \propto \exp\left[-\sum_i \frac{\left(H_i - H(z_i)\right)^2}{2\sigma_i^2}\right]. \tag{10.44}$$

The expression $H(z_i)$ is characterized by Equation (10.44), which is calculated for each redshift z_i. On the other hand, H_i denotes the measured value of the Hubble parameter at z_i, while σ_i signifies the uncertainty associated with the i th measurement.

The only available information regarding the free parameters indicates that each component of the Universe, $\Omega_{i,0}$, must comply with the condition $0 \leq \Omega_{i,0} \leq 1$. The prior distribution for the current Hubble parameter H_0 can be established based on the existing data. A widely accepted choice for this prior is a uniform distribution (flat prior) within defined limits.

$$\Omega_{m,0} \in [0,1],\ H_0 \in [20,100],\ \Omega_{\ ,0} \in [0,1],\ w_0 \in [-3,1],\ w_a \in [-3,2]. \tag{10.45}$$

Up to this point, three essential steps in the process of parameter inference for a specific model have been completed. First, observational data are collected to facilitate comparison with the model parameters. Second, the likelihood function is constructed in accordance with the underlying theoretical framework. The functional dependence of the likelihood on the model parameters varies with the type of observational data used. Finally, numerical methods are employed to perform the parameter inference.

When combining various datasets, it is important to remember that different models may yield varying results. This highlights the need for caution when developing new models; that is, the conclusions do not diverge significantly from the standard ΛCDM model analysis.

The results corresponding to the estimated parameters are summarized in Table 10.1. The error bar plots for both models are shown in Figures 10.1 and 10.2, which depict how the model fits with the data compared to ΛCDM. Figure 10.3 displays the contour plots

for the extracted parameters of Model A, obtained from both datasets. In addition, Figure 10.4 provides a visual representation of the contour plots associated with the parameters of Model B.

A significant deviation from the established Planck values is observed in both scenarios when examining the inferred parameter values. This deviation indicates that the supernova data exert a dominant influence on the results, as the effect is notably pronounced.

To enhance the robustness of our findings, it will be essential to incorporate additional datasets, including CCs, Planck, and several others, strengthening the overall analysis and conclusions drawn from our study.

With regard to the χ^2 values, Model A yields a value of 1420.3, while Model B results in a slightly lower value of 1418.69.

Table 10.1 68% Confidence-Level Constraints on the Two Considered Models

Model	$\Omega_{m,0}$	H_0	$\Omega_{\Lambda,0}$	w_0	w_a
Model A	0.310 ± 0.012	73.506 ± 6.7	0.689 ± 0.012	—	—
Model B	0.307 ± 0.023	72.75 ± 7.5	—	-0.86 ± 0.072	-0.49 ± 0.58

We have considered BAO + SN to obtain a joint best-fits.

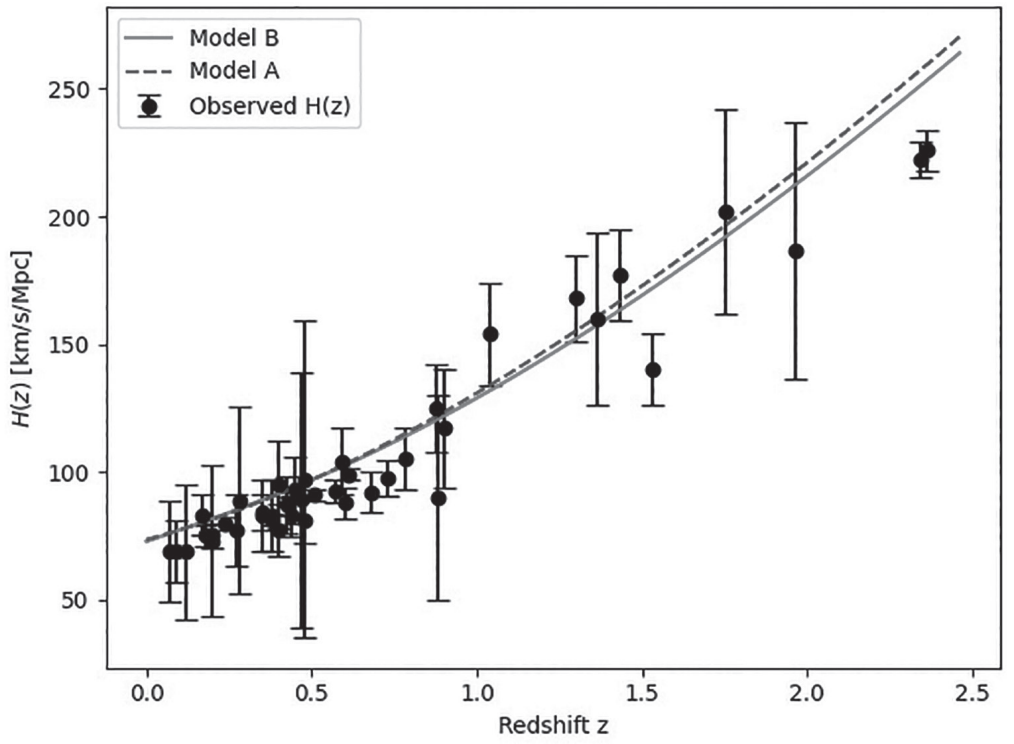


Figure 10.1 The evolution of the Hubble function versus redshift z for CC data. The error bars are the observed $H(z)$ values from the data.

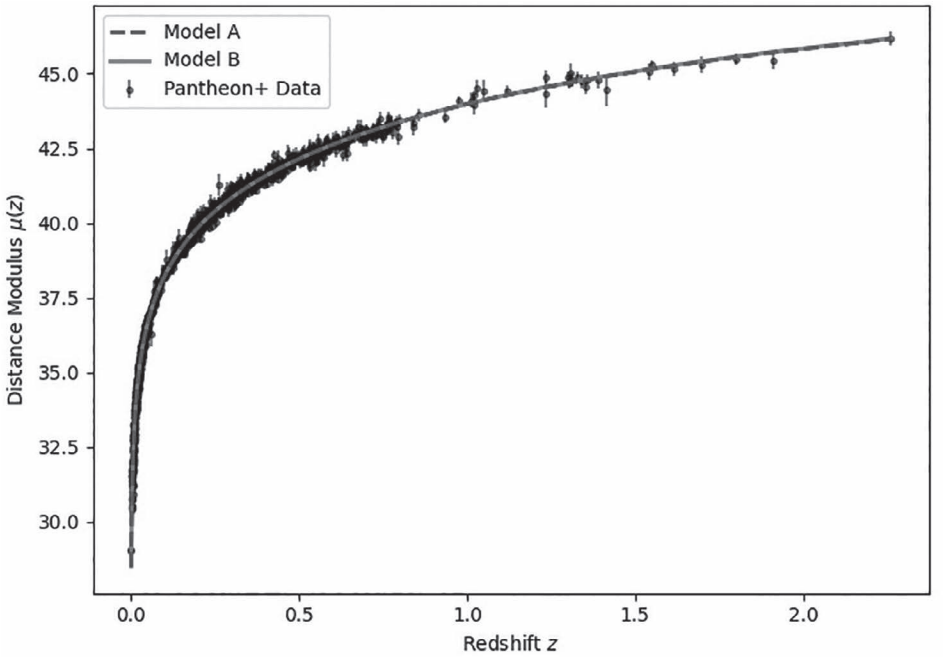


Figure 10.2 The evolution of the $\mu(z)$ as a function of redshift z for Pantheon+ data. The error bars are the data's observed $\mu(z)$ value.

10.6 MACHINE LEARNING BACKGROUND

Machine learning, which is a subset of artificial intelligence, focuses on the mathematical modeling of data. It reveals concealed patterns within datasets by persistently reducing a loss function until an ideal set of model parameters is achieved. In recent years, ANNs have become some of the most powerful models in machine learning. These networks contain thousands to millions of parameters, known as weights, enabling them to approximate highly complex nonlinear functions (Hornik et al., 1990).

Determining the optimal set of hyperparameters—essentially, the ideal neural network design—is a challenging task. Traditionally, this involves testing multiple hyperparameter combinations to identify the most effective one. However, with the advancement in computational capabilities, numerous optimization and parameter estimation strategies have been employed to streamline this process. Notably, metaheuristic optimization methods, such as genetic algorithms, have proven useful in solving these optimization problems without relying on gradient information. This segment provides a concise overview of ANNs and their algorithms.

10.6.1 Artificial Neural Networks

ANNs are widely utilized across numerous scientific disciplines due to their capacity to handle extensive and intricate datasets. According to the universal approximation theorem, ANNs have the theoretical ability to represent any nonlinear function (Hornik et al., 1990), which makes them particularly effective for analyzing data with unknown and often

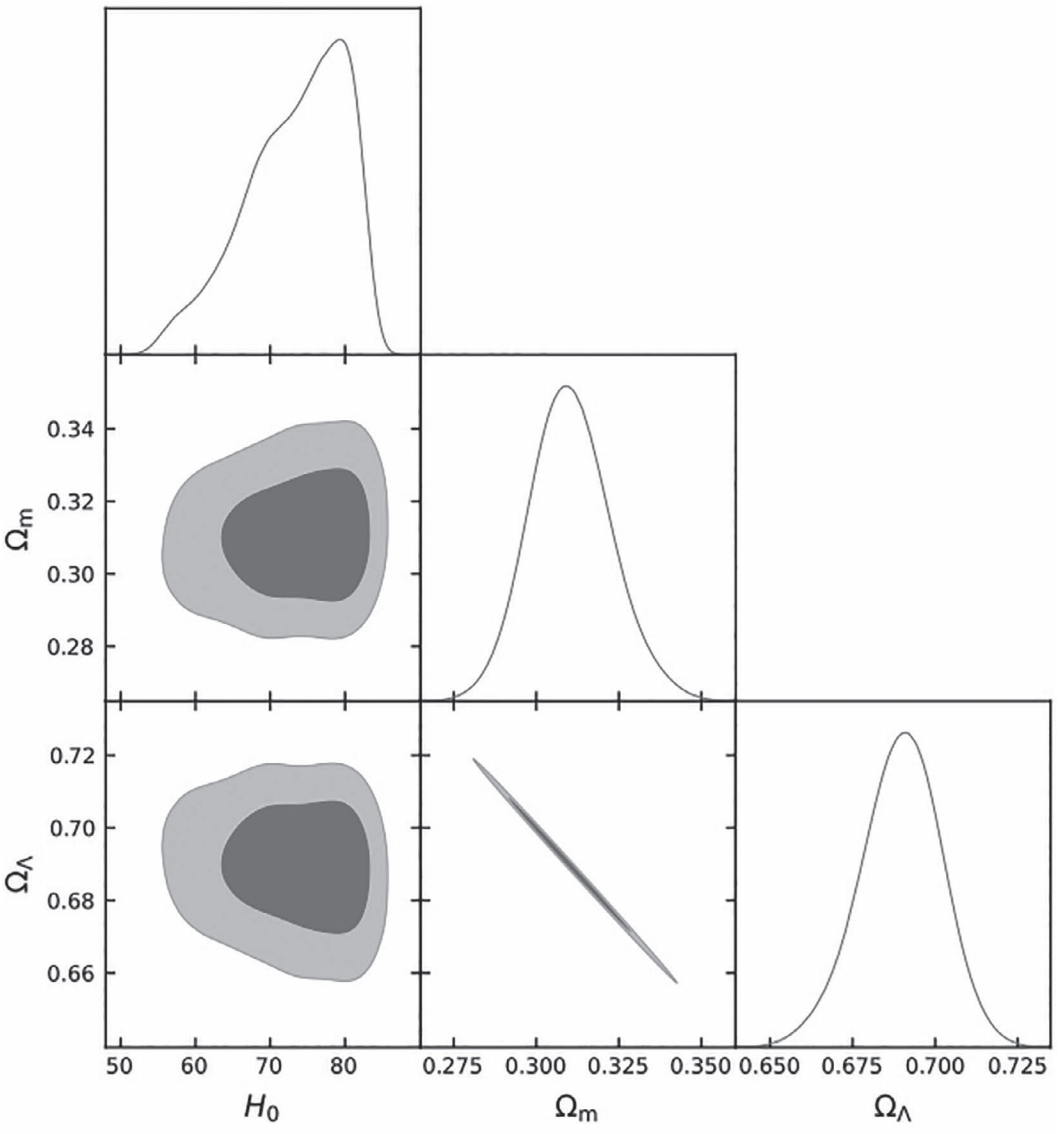


Figure 10.3 The 2D posterior distributions with $1-2\sigma$ confidence regions for model A.

multidimensional relationships between variables. A comprehensive treatment of neural networks falls outside the scope of this work. However, extensive literature exists for readers seeking an in-depth understanding (Bishop and Nasrabadi, 2006; Nielsen, 2015). For those interested in foundational concepts, core algorithms, and their use in cosmology, refer to de Dios Rojas Olvera et al. (2022) and Chantada et al. (2023, 2024).

ANNs are computational frameworks inspired by the structure of biological neural systems. They model the functioning of synapses using layers of interconnected units called neurons or nodes, which are the essential elements for processing information. The most basic structure of an ANN is the feed-forward neural network, which consists of three main types of layers: an input layer that receives the data, one or more hidden layers

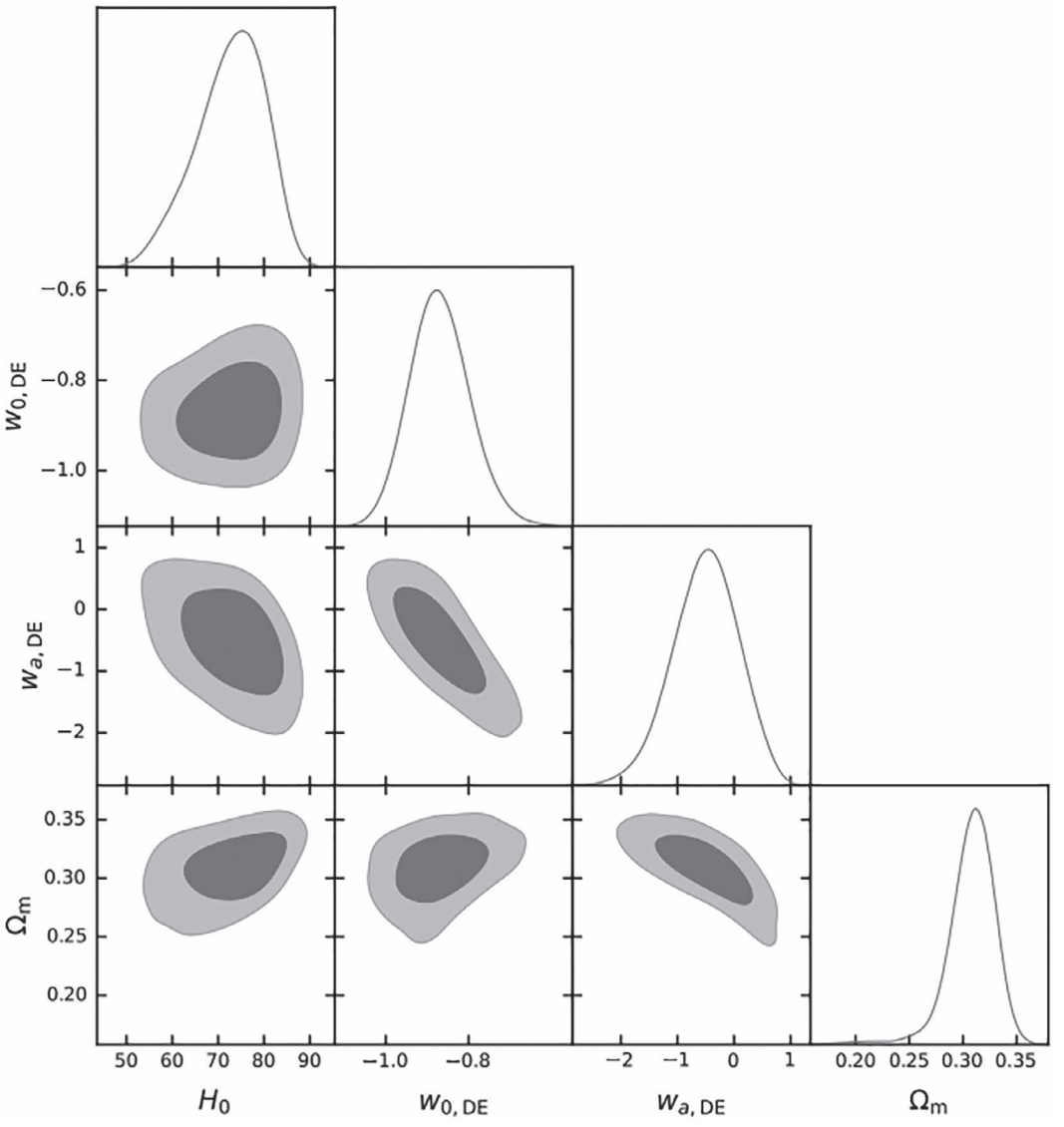


Figure 10.4 The 2D posterior distributions with $1-2\sigma$ confidence regions for model B.

that identify patterns and add nonlinearity, and an output layer that generates the final prediction.

In contrast to the model's weights, which are adjusted during training, hyperparameters must be specified before the training process starts. This encompasses factors such as the number of layers, the count of neurons within each layer, the total number of training epochs, and the selection of the activation function. Furthermore, because training typically relies on gradient descent and backpropagation methods (Nielsen, 2015), additional parameters like the learning rate and batch size are also regarded as hyperparameters. Some of these are predetermined in real-world applications, while others are optimized using hyperparameter tuning techniques.

10.6.2 Methodology

A brief overview of the ANN methodology employed in this study for reconstructing cosmic expansion data is presented in the following text. An ordinary ANN consists of an input layer, multiple hidden layers, and an output layer, with each layer made up of neuron processing units. The input layer receives the features from the dataset, which serve as initial values for the neurons in that layer. On the other hand, the output layer generates the ultimate predictions, which are assessed using a loss function that measures the discrepancy between the network's output and the true target values.

The overall structure of the network is defined by these three layers: input, hidden, and output, which direct the information flow from input to output. This layout ensures the signals progress through the network in a specific direction. As an illustration, Figure 10.5 presents a basic ANN architecture that includes a hidden layer with n_k neurons. Each redshift input in the network leads to the prediction of a general cosmological quantity $\alpha(z)$ and its associated uncertainty $\sigma_\alpha(z)$. The ANN operates by performing a sequence of transformations, starting with a linear combination of the inputs (via weights and biases), followed by a nonlinear activation function in each layer. The computations are performed sequentially through each layer, culminating in a final linear transformation at the output layer. As a result, each input passes systematically through the entire network to produce the desired output.

10.7 IMPLEMENTATION OF ANN

To solve differential equations with ANNs, the problem is approached as an optimization problem. The goal is to find the internal parameters of the network that reduce a cost (or loss) function L, which measures how closely the ANN's output aligns with the specified differential equation. Furthermore, the solution must comply with the initial and/or boundary conditions of the system. The loss function is usually expressed as follows:

$$L(v_M, t) = \sum_{i=1}^{N} K_i(v_M, t)^2. \tag{10.46}$$

Here $v_M = v_M(t) = \left\{v_M^1(t), v_M^2(t), \ldots, v_M^N(t)\right\}$ denote the vector of outputs generated by the ANN, which represents each of the N dependent variables at a specific time t (the independent variable). The superscript indicates the label of each dependent variable. In this context, K_i denotes the residual associated with the i th equation in a system of N differential equations. While the exact solution would satisfy the differential equations, the ANN solution generally does not. In contrast, assessing the i th equation using the ANN output v_M yields a value that is not zero, known as the residual of that equation. Specifically, an architecture is employed in which each dependent variable is modeled by a separate ANN. Each $v_M^j(t)$ represents the output of a separate ANN, with each network receiving a single input.

Although Equation (10.46) effectively evaluates the performance of ANNs in solving the equations of a differential system, it fails to account for the initial and boundary conditions. To address this, a reparametrization $\tilde{v}(v_M, t)$ of v_M can be introduced, which ensures that these conditions are met. For example, in a first-order ordinary differential equation (ODE)

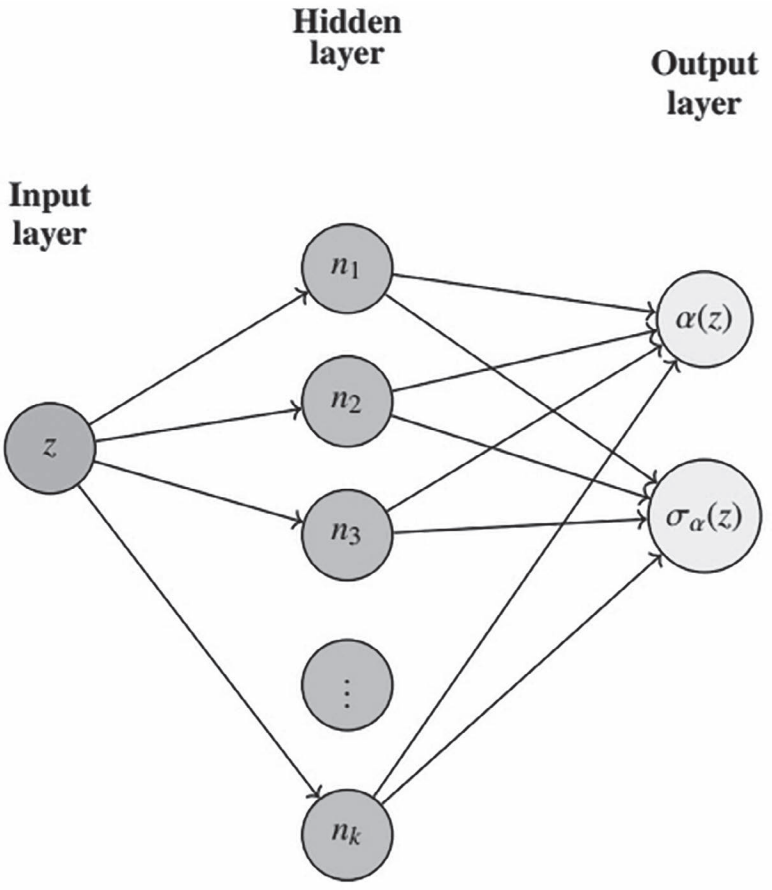


Figure 10.5 Schematic representation of the ANN architecture, where the input is the redshift z of a cosmological parameter $\alpha(z)$, and the outputs correspond to its predicted value and associated uncertainty.

system with initial conditions $v(t_0) = v_0$, the reparametrization needs to fulfill $\tilde{v}(v_M, t_0) = v_0$. A suggested reparametrization, detailed in Mattheakis et al. (2022), is:

$$\tilde{v}(t) = v_0 + \left(1 - e^{-(t-t_0)}\right) v_M(t). \tag{10.47}$$

With this adjustment, the loss function to minimize for solving ODEs becomes:

$$L(\tilde{v}, t) = \sum_{i=1}^{N} K_i(\tilde{v}, t)^2. \tag{10.48}$$

The loss function presented in Equation (10.48) utilizes the reparametrized outputs from the ANNs, generating a scalar that measures the accuracy with which the ANNs address the ODE system at each moment t.

A modified approach, as discussed in Flamant et al. (2020), enables trained neural networks to represent a collection or bundle of solutions for a specific system of differential equations across a continuous range of parameters $\Theta = \{\theta_1, \theta_2, \ldots, \theta_p\}$. To implement this method, the neural network v_M must incorporate p additional inputs that correspond to the parameters of the system. Therefore, if t_0 is not one of the system's parameters, the formulation of Equation (10.47) is adjusted accordingly.

$$\tilde{v}(t,\Theta) = v_0(\Theta) + \left(1 - e^{-(t-t_0)}\right) v_M(t,\Theta). \tag{10.49}$$

Since the system's parameters can include the initial conditions, v_0 is now considered a function of Θ. Furthermore, Equation (10.42) is modified to:

$$L(\tilde{v},t,\Theta) = \sum_{i=1}^{N} K_i(\tilde{v},t,\Theta)^2 \tag{10.50}$$

This denotes the loss function for the array of solutions at time t for the differential system characterized by the parameters Θ.

10.7.1 Statistical Analysis

The preceding section outlined the process of training ANNs to solve systems of differential equations. The following section describes how this technique can be integrated into a computational pipeline for statistical analysis, enabling the evaluation of theoretical models against observational data.

- **Data and Model Setup:** Consider a dataset $\mathcal{D}$ and a model $M(\Theta)$, where Θ represents the vector that includes all parameters and observables of the model that can be checked against the data $\mathcal{D}$. The objective is to make inferences about the possible values of Θ using the dataset $\mathcal{D}$.
- **Solving the Differential System:** If the objective involves solving a differential system embedded in the model, the ANN method proves to be a valuable approach. This differential system is defined by its equations, which also incorporate initial and/or boundary conditions.
- **Focusing on ODEs:** In particular, as discussed earlier in this section, we will concentrate on systems where the differential equations are ODEs. The system's equations can be expressed through the residuals $K_i(v,t,\Theta)$, where $v = \{v_1, v_2, \ldots, v_N\}$ denotes the dependent variables of the system.
- **Training the ANN:** With the differential system properly defined, the first key step in the process, training the ANN, can now be undertaken.
- **Setting the Loss Function and Reparametrization:** At this stage, the loss function $L(\tilde{v},t,\Theta)$ as defined in Equation (10.50) is introduced. Along with the loss function, a reparametrization $\tilde{v}(t,\Theta)$ is defined to enforce the boundary and initial conditions on the ANN outputs, ensuring that the network respects the necessary conditions for the system.
- **Training the ANN:** Once the loss function and reparametrization are established, the training of the ANN is initiated. The process continues until the loss function converges to a sufficiently low value, deemed acceptable based on predefined criteria, at which point training is terminated.

- **Using the Trained Network:** After training is completed, the ANN, in combination with the reparametrization $\tilde{v}(t,\Theta)$, can now be used to generate bundle solutions for the dependent variables of the ODE system. These solutions are denoted as $v(t,\Theta)$.
- **Statistical Analysis:** After acquiring the solutions from the trained ANN, the subsequent step is to apply them in a statistical analysis in conjunction with the observational data. This enables us to make inferences about the values of the model parameters Θ.
- **MCMC and Posterior Distribution:** In this scenario, the statistical analysis is carried out by performing MCMC using the likelihood $L(\mathcal{D}|\Theta)$. This procedure produces samples from the posterior distribution $P(\Theta|\mathcal{D})$, which are then utilized for parameter estimation.

The procedure described previously is illustrated in Figure 10.6, where the differential system is characterized by a single ODE, and the statistical evaluation is conducted using MCMC sampling.

10.8 CONCLUSION

The substantial rise in cosmological observations in recent years has led to a deeper understanding of the Universe. Consequently, Bayesian statistics has gained interest as a powerful alternative approach. This chapter provides an overview of Bayesian statistics and demonstrates its application in cosmology through various examples.

As advanced statistical methods are increasingly employed to maximize the value of data, it is crucial to ensure these techniques are implemented and used properly. Gathering data is expensive and challenging, and statistical methods play a vital role in maximizing the utility of the available information. However, it is important to remain vigilant about systematic errors, as improper data handling can lead to invalid conclusions. There will always be limitations to what the statistical power of a given dataset can reveal.

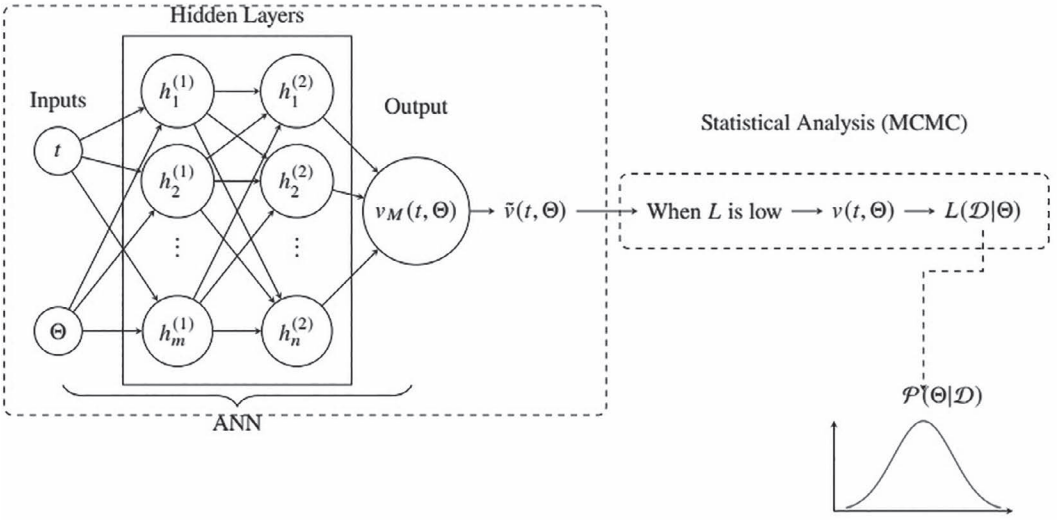


Figure 10.6 A demonstration of applying ANNs to solve a differential system within a model using bundle methods.

Calculating the posterior distribution is a resource-intensive process that involves examining the likelihood and prior over multiple dimensions. MCMC techniques, particularly the MHA, were used to navigate the cosmological parameter space. A Markov process is a stochastic process in which the likelihood of future states depends solely on the present state, not on previous ones. Any algorithm that relies on random number generation is classified as a Monte Carlo method. In Metropolis–Hastings, a transition kernel is employed to generate a series of points, forming a chain within the parameter space, which enables the approximation of the posterior distribution. The elements of the Markov Chain are generated probabilistically based on the specific algorithm utilized. While the MHA is a straightforward tool for Bayesian inference, other samplers were introduced to more efficiently explore complex posterior distributions.

In conclusion, Bayesian statistics plays an essential role in cosmology, enabling the identification of model parameters that most accurately represent the Universe. The standard cosmological model, ΛCDM , was assessed in relation to current observational data such as CC, BAO, and SN. However, this does not imply that the ΛCDM model is the definitive choice. Rather, it indicates that, based on the existing datasets, this model provides a better fit compared to alternative models, although further examination and additional data are needed to draw a definitive conclusion.

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Chapter 11

Progress in Lattice-Based Digital Signature

Systematic Review, Challenges, and Research Directions

Pooja Bhatt and Rachna Navalakhe

11.1 INTRODUCTION

As the scientists predicted, the quantum computer is the near future of the digital world; supercomputers can be assumed as one of the biggest threats to universal security and cryptographic algorithms used for digital security. Lattice-based cryptography is considered the most favorable post-quantum cryptography (PQC) scheme against quantum cryptography. Lattice-based digital signature is one of the extensively considered post-quantum cryptographic schemes due to its straightforward arithmetical operations, wealthy mathematical roots, and worst-case security.

The rapid advancement of quantum computing poses significant challenges to classical cryptographic systems. Algorithms such as RSA and elliptic curve cryptography (ECC), which rely on the computational difficulty of problems like integer factorization and discrete logarithms, are increasingly vulnerable in the face of quantum algorithms, most notably, Shor's algorithm. Shor's ability to solve these hard problems in polynomial time threatens the foundational security assumptions of widely deployed public-key cryptography. This looming threat has spurred global efforts toward developing cryptographic primitives that can withstand quantum adversaries, a movement known as PQC.

Among the various post-quantum approaches, lattice-based cryptography has emerged as one of the most promising and mathematically grounded solutions. The security of lattice-based schemes relies on the hardness of problems such as short-integer solution (SIS) and learning with errors (LWE)—problems for which no efficient quantum algorithms are currently known. Lattice-based constructions offer several advantages: worst-case to average-case hardness reductions, strong security guarantees, and the potential for efficient implementation. These features make lattice-based cryptography a compelling candidate for post-quantum digital signature schemes (DSSs).

Recent years have witnessed the development of robust and practical lattice-based DSSs. In 1979, a mathematician, Leslie Lamport, invented the world's first signature based on a hash function (Lamport 1979). Lamport found that, by using simple hash functions or one-way functions, it was possible to build very powerful digital signature systems. To address the inability to sign multiple messages with a single key, Ralph Merkle proposed a new DSS based on Merkle's tree (Merkle 1980). Later, Robert Winternitz proposed a further upgrade of DSS (Winternitz 1984) based on the Merkle method described earlier. In practice, this approach reduces the signature and public-key size by a factor of four to eight, at the cost of increasing the time it takes to sign and verify. In the 1980s, Goldreich and Levin (1989) pointed out that there is a way to create a signature that does not need to be maintained. A short "verification tree" of one-time public keys is generated instead of generating all keys upfront (Liu et al. 2024).

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Shor's algorithm (Shor 1999) was implemented for the factorization of large integers and for the solution of problems related to discrete algorithms, which was established on a polynomial-time quantum algorithm proposed by Shor. As the advent of quantum computers is considered imminent, cryptographic schemes based on hard number-theoretic problems—particularly those relying on the discrete logarithm problem—are becoming vulnerable. This affects most widely used public-key encryption schemes, including RSA (Rivest et al., 1978), DSA (National Institute of Standards and Technology, 2009), DSS, and ECC (Breuil et al., 2001). After Shor's quantum algorithm, Oded Regev has put forward a scheme (Regev 2025) that actually lessens the pace required for the factorization of hugely large numbers, which is considered a considerable refinement of Shor's quantum algorithm. The age of quantum computing is predicted to be nearer; this increased the need for PQC due to its mathematically complex computational problems, which led to more security. The hard problems like closest vector problem (CVP) and shortest vector problem (SVP) of lattice-based cryptography, which worked as aversion to quantum cryptography (Ajtai et al. 2001; Dinur et al. 1998), resulted in lattice-based cryptography as a weapon against quantum computing.

Miller (1985) and Koblitz (1987) introduced ECC (Singh et al. 2020, 2023), but it took 20 years to come in the form of secured system. The public-key DSS (Diffie and Hellman 2022) is defined as a process of recognizing information digitally like a normal, manual handwritten signature on a piece of paper. This idea was initiated by Diffie and Hellman (2022) on the conceptual suggestion given by Rivest et al. (1978).

There were many scholars or researchers who proffered several signature schemes to achieve the security of signatures for getting ensured properties of digital signatures in the past few decades. DSSs include RSA (Rivest et al. 1978; ElGamal 1985; Fiat and Shamir 1986; Guillou and Quisquater 1990; Ong and Schnorr 1991) and DES/DSA (NIST 2009; Washington 2008; Diffie and Hellman 2022).

Apart from standard lattice structures, we frequently sightsee schemes based on special lattices, for example, "mathematical theory of post-quantum cryptography" (Zheng et al. 2023). These lattice structures recurrently bid robust security assumptions along with shorter public keys or signatures.

All the above schemes were constructed on a public-key encryption scheme, but they resist classical computer attacks only from an ultra-security point of view. We need some more progressive lattice-based DSSs to resist the attack from quantum computers.

Nowadays, lattice-based cryptography is trending in technical research fields. The National Institute of Standards and Technology (NIST) is an organization of the Commerce Department under the US government. The process of PQC standardization is initiated by NIST in 2016, with lattice-based cryptography as a significant element. Five schemes were submitted for the NIST PQC in Round 1, which were lattice-based signature schemes. These were, namely, DRS, pq-NTRU sign, q Tesla, CRYSTALS-Dilithium, and Falcon. The DRS scheme was based on the diagonal dominant basis in a hidden structured lattice (Plantard et al. 2018). The last two signature schemes, CRYSTALS-Dilithium (Ducas et al. 2018) and Falcon (Fouque et al. 2018), survived after Round 3. Both were based on rigorous computational problems over module lattices (Langlois and Stehlé 2015).

This chapter explores lattice-based digital signatures as a reliable and quantum-resilient alternative to classical algorithms. As traditional cryptographic systems face growing threats from quantum computing (Bhatt and Navalakhe 2025), lattice-based approaches present a mathematically sound and practically viable solution for the next generation of secure communication (Navalakhe and Atre 2023; Bhatt and Navalakhe 2023)

Objectives of the Chapter

The objectives of the chapter are:

1. To present a comprehensive review regarding the development of DSSs built on lattice-based cryptographic principles to understand how these schemes evolved from theoretical models to practical, real-world systems, especially in the context of PQC;
2. To explain the underlying mathematical foundations to clearly describe the hard lattice problems—such as SIS and LWE—that form the backbone of these signature schemes, ensuring that readers can understand both the theory and its cryptographic significance;
3. To analyze and compare major lattice-based signature schemes like CRYSTALS-Dilithium, Falcon, and Bimodal Lattice Signature Scheme (BLISS), examining key metrics such as "key sizes", "signature lengths", "computational efficiency", "security guarantees", and "implementation complexity";
4. To identify and discuss current technical challenges that lattice-based signatures face today, including side-channel vulnerabilities, optimization trade-offs between performance and security, and the difficulty of parameter selection;
5. To suggest potential research directions by identifying open problems and limitations in current systems, to inspire new research efforts aimed at improving the efficiency, robustness, and usability of lattice-based digital signatures;
6. And, to support the broader adoption of PQC toward the global movement of securing digital communications against quantum threats by promoting a deeper understanding of lattice-based solutions.

Organization of the Chapter

The rest of the chapter is organized as follows. Section 11.2 enlightens preliminaries. Section 11.3 discusses state-of-the-art lattice-based signatures. Section 11.4 highlights specific digital signatures. Section 11.5 focuses on challenges and issues in lattice-based digital signatures. And, finally, Section 11.6 concludes the chapter with future scope.

11.2 PRELIMINARIES

11.2.1 Lattice-Based Cryptography

- **Lattices:** In Euclidean space, a regular repeated arrangement of points is called a lattice, where each point is generated by a linear combination of a set of vectors of basis with integer coefficients. We can allude to Zheng et al. (2023) for more attributes of the lattice.

11.2.1.1 Mathematical Representation

"A lattice is a discrete additive subgroup of a vector space that maintains a minimum distance between its points. Given a set of linearly independent vectors $\mathbf{b}_1, \mathbf{b}_2 \ldots \mathbf{b}_n \in R^m$, these vectors form the basis of a lattice L. The lattice L generated by these basis vectors is defined as:

$$L(\mathbf{b}_1, \mathbf{b}_2 \ldots \mathbf{b}_n) = \left\{ \sum_{i=1}^{n} \mathbf{a}_i \mathbf{b}_i : \mathbf{a}_i \epsilon \mathbf{Z} \right\}$$

Here, the vectors $\mathbf{b}_1, \mathbf{b}_2 \ldots \mathbf{b}_n$ are considered column vectors, meaning the basis can be represented as a matrix B. The integers n and m refer to the rank and dimension of the lattice, respectively. The minimum distance λ of the lattice is defined as the length of the shortest nonzero vector x in the lattice; that is,

$\lambda = \mathbf{min}\left\{\|\mathbf{x}\| \mathbf{x} \in \mathbf{L}, \mathbf{x} \neq \mathbf{0}\right\}$ " (Liu et al. 2024). For the clear visual intuition Figure 11.1 represents Lattice with color variations for both nodes and lines.

- **Lattice Problems**: Lattice-based cryptography depends on the complexity of certain problems defined over lattices. The most common ones comprise:
  - **SVP**: Finding the shortest non-zero vector in a lattice basis B.

11.2.1.2 Mathematical Representation

Let a lattice $L \subset R^n$, SVP instructs to find the non-zero vector in L that has the shortest Euclidean length.

$$\text{That is, find } \mathbf{v} \in \mathbf{L} \setminus \{\mathbf{0}\} \mathbf{such that} \|\mathbf{v}\| = \min_{\mathbf{u} \in \mathbf{L} \setminus \{\mathbf{0}\}} \|\mathbf{u}\|$$

Here, $\|.\|$ is generally the Euclidean norm, that is, $\|v\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$

SVP is hard for the following reasons:

- In higher dimensions, the number of vectors grows exponentially.
- There is no known polynomial-time algorithm to solve SVP exactly.

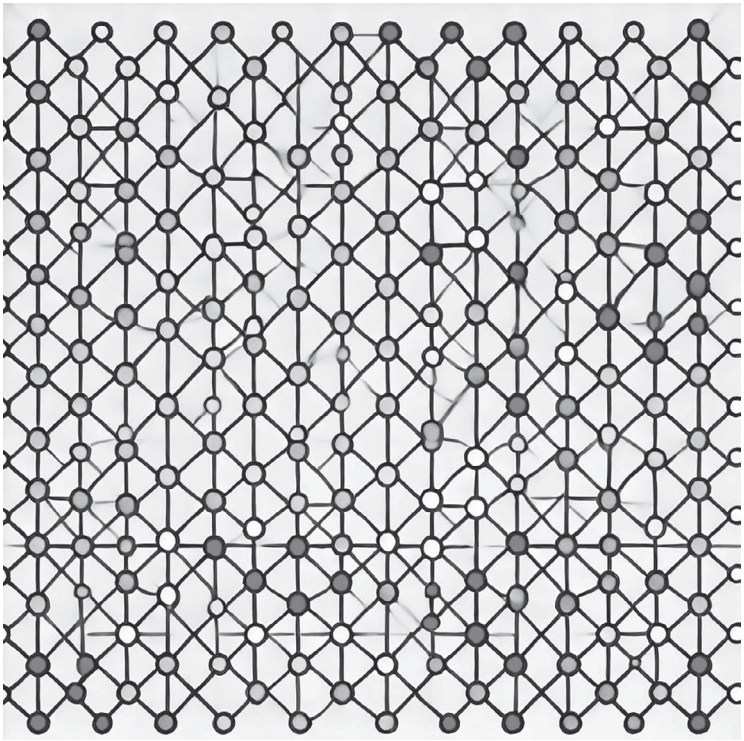


Figure 11.1 Lattice with color variations for both nodes and lines.

- The problem is NP-hard under randomized reductions.
- Many cryptographic systems rely on the hardness of SVP (like lattice-based encryption and signatures).

11.2.1.2.1 2D Example of SVP

Consider the lattice L is engendered by:

$$\mathbf{b_1} = (\mathbf{4},\mathbf{1}), \mathbf{b_2} = (\mathbf{1},\mathbf{3})$$

Then the lattice points are:

$$\mathbf{v} = \mathbf{z_1}\mathbf{b_1} + \mathbf{z_2}\mathbf{b_2}\,\mathbf{where}\,\mathbf{z_1},\mathbf{z_2} \in \mathbf{Z}$$

We calculate:

- $\|\mathbf{b_1}\| = \sqrt{\mathbf{4^2} + \mathbf{1^2}} = \sqrt{\mathbf{17}}$
- $\|\mathbf{b_2}\| = \sqrt{\mathbf{1^2} + \mathbf{3^2}} = \sqrt{\mathbf{10}}$

But there might exist a combination, like $\mathbf{b_1} - \mathbf{b_2} = (\mathbf{3},-\mathbf{2})$, that is even shorter:

$$\|\mathbf{b_1} - \mathbf{b_2}\| = \|\ (\mathbf{3},-\mathbf{2})\ \| = \sqrt{\mathbf{3^2} + (-\mathbf{2})^2} = \sqrt{\mathbf{13}}$$

So, solving SVP means checking all non-zero integer combinations to find the shortest one—a hard task in higher dimensions. Figure 11.2 represents shortest vector problem visualization in 2D.

- **LWE**: Solving a cryptographic problem comprises a system of linear equations that is disturbed by minute random noise. LWE depends on the learning framework, but generally, it refers to situations where the learned function does not perfectly match the target function, meaning some mistakes or approximation errors are allowed.

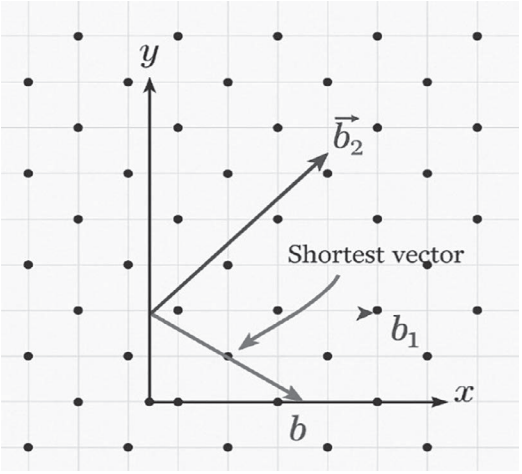


Figure 11.2 Shortest vector problem visualization in 2D.

11.2.1.3 Mathematical Representation

Consider:

- $\mathbf{s} \in \mathbf{Z}_q^n$: called secret vector
- $\mathbf{A} \in \mathbf{Z}_q^{m \times n}$: called known matrix (public)
- $\mathbf{e} \in \mathbf{Z}_q^m$: called small error vector (noise)
- $\mathbf{b} = \mathbf{As} + \mathbf{e} \bmod \mathbf{q}$: called output vector

The objective of the problem is to determine the secret vector **s** from the given parameters. However, because of the inclusion of an error vector **e**, the resulting system is not an exact linear system but a noisy one.

LWE is hard for the following reasons:

- Due to error **e**, we need to deduce through noise—and that is why it becomes as hard as solving worst-case lattice problems, like the SVP, as shown in Figure 11.3 Learning with errors.

Example:
Consider:

- $\mathbf{q} = 7$
- $\mathbf{s} = [3, 4]$
- $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 1 & 6 \\ 2 & 3 \end{bmatrix}$
- $\mathbf{e} = [1, -1, 0]$

Calculate:

$$\mathbf{b} = \mathbf{As} + \mathbf{e} \bmod 7 \quad \mathbf{b} = \begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix}$$

We get public data:

$$\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 1 & 6 \\ 2 & 3 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix}$$

We need to deduce $\mathbf{s} = [3, 4]$ from the given public data, which is not easy. It defines the hardness of the problem LWE.

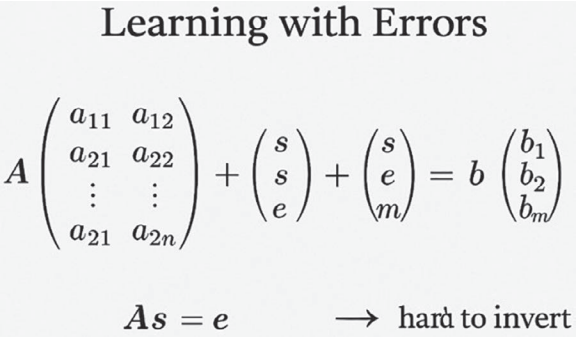


Figure 11.3 Learning with errors.

- **Shortest Independent Vectors Problem (SIVP):** Given a basis $\mathbf{B} \in \mathbf{Z}^{n \times n}$ of a lattice **L**, the objective is to find n linearly independent lattice vectors $\mathbf{S} = [\mathbf{s_1}, \ldots, \mathbf{s_n}]$ (where $\mathbf{s_i} \in \mathbf{L}(\mathbf{B})$ for all i) such that the maximum length among them, $\|\mathbf{S}\| = \mathbf{max_i} \|\mathbf{S}\|$ is minimized.

In lattice-based cryptography, we often deal with approximate versions of certain problems, indicated by a subscript γ representing the approximation factor. For example, in the SVP with approximation factor γ (SVP_γ), the objective is to find a vector whose length is at most times that of the shortest non-zero vector in the lattice as shown in Figure 11.4 Shortest independent vector problem.

It is important to note that these approximations can be defined using various norms, but they are most commonly considered in terms of the Euclidean norm $\|\mathbf{x}\| = \sqrt{\sum_i \mathbf{x}^2}$ (Liu et al. 2024).

11.2.1.4 Mathematical Representation

Consider:

- **L** be a lattice of rank **n** in $\mathbf{R}^n$
- $\lambda_n(\mathbf{L})$ be the $\mathbf{n}^{th}$ successive minimum, defined as:

$$\lambda_n(\mathbf{L}) = \mathbf{min}\{\mathbf{r} \in \mathbf{R} \mid \exists \mathbf{lineralyindependent} \mathbf{v_1}, \ldots \mathbf{v_n} \in \mathbf{L} \mathbf{with} \|\mathbf{v_i}\| \leq \mathbf{r}\}$$

The SIVP is to find **n** linearly independent $\mathbf{v_1}, \ldots \mathbf{v_n} \in \mathbf{L}$, such that:

$$\max_{i=1}^{n} \|\mathbf{v_i}\| \leq \gamma . \lambda_n(\mathbf{L})$$

where $\gamma \geq 1$ is the approximation factor (exact if $\gamma = 1$).

2D Example:

Consider the lattice is generated by:

- $\mathbf{b_1} = (4, 0)$
- $\mathbf{b_2} = (1, 3)$

We want to find two short lattice vectors $\mathbf{v_1}, \mathbf{v_2} \in \mathbf{L}$, such that they are linearly independent and short.

Try these combinations:

- $\mathbf{v_1} = \mathbf{b_2} = (\mathbf{1}, \mathbf{3})$
- $\mathbf{v_2} = \mathbf{b_1} - \mathbf{b_2} = (\mathbf{3}, -\mathbf{3})$

Lengths:

- $\|\mathbf{v_1}\| = \sqrt{\mathbf{1^2} + \mathbf{3^2}} = \sqrt{\mathbf{10}}$
- $\|\mathbf{v_2}\| = \sqrt{\mathbf{3^2} + (-\mathbf{3})^2} = \sqrt{\mathbf{18}}$

$\mathbf{v_1}$ **and** $\mathbf{v_2}$ are short as well as form a basis, so this solves a version of SIVP.
SIVP is hard for the following reasons:

- SIVP is at least as hard as the SVP.
- Finding multiple short, independent vectors is more complex than finding just one.
- Quantum- and classical-hardness results link SIVP to cryptography (e.g., in LWE).

11.2.2 Digital Signatures

- **Definition:** A digital signature is a PQC scheme that permits an entity to verify the authenticity and integrity of a message or document.

Process: A digital signature involves two main operations, as shown in Figure 11.5 Digital signature process:

i. **Signing:** A user signs a message using their private key, creating a signature.
ii. **Verification:** Anyone can verify the signature using the matching public key to ensure that the message was signed by the legitimate signer and has not been tampered with.

- **Requirements:** A DSS necessarily satisfies several requirements:
  i. **Unforgeability:** It should be computationally infeasible for an adversary to create a valid signature without having any idea about the private key.
  ii. **Integrity:** The signature must verify the integrity of the signed message.

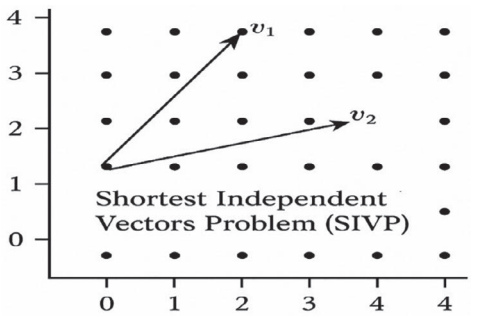


Figure 11.4 Shortest independent vector problem.

11.2.2.1 Mathematical Framework

Digital signature mainly includes three algorithms:

i. **Key Generation**
   Generates a pair:
   - **Private key:** sk
   - **Public key:** pk

ii. **Signing**
   To sign a message **m**, compute a signature **σ** using:

$$\sigma = \mathbf{Sign}_{sk}(m)$$

iii. **Verification**
   To verify a signature:

$$\mathbf{Verify}_{pk}(m, \tilde{A}) \rightarrow \mathbf{trueorfalse}$$

Example: RSA Digital Signature

Key Generation:

- Pick primes: $p = 11, q = 13 \rightarrow N = pq = 143$
- Calculate $\phi(N) = (p-1)(q-1) = 120$
- Select $e = 7$, $d = 103$ such that $ed \equiv 1 \bmod 120$
- **Public key:** $(N = 143, e = 7)$
- **Private key:** $d = 103$

Signing:

Given message $m = 9$, the signature is:

$$\sigma = m^d \bmod N = 9^{103} \bmod 143 = 48$$

Verification:

Check if:

$$\sigma^e \bmod N = 48^7 \bmod 143 = 9 = m$$

✅ Signature is valid.

11.2.3 Lattice-Based Digital Signatures

- **Definition:** A lattice-based DSS is a PQC signature scheme that relies on the complexity of lattice problems, particularly SVP or LWE, to ensure security (Figure 11.6 visualises the process of Lattice-Based Digital Signatures).
- **Key Features:**
  i. **Security Assumptions:** The security of lattice-based signatures relies on the hardness of solving lattice-based problems, which is considered to be hard even for quantum computers.

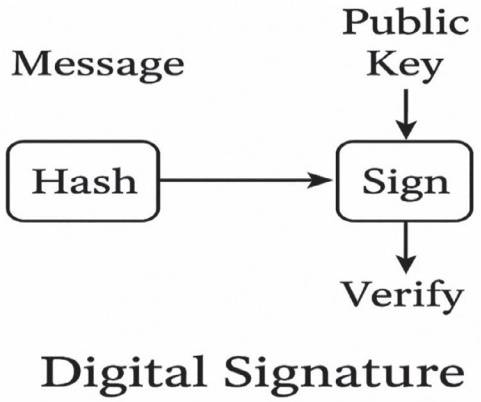


Figure 11.5 Digital signature process.

ii. **Signature Generation:** Lattice-based signatures typically involve creating signatures by leveraging properties of lattice structures and ensuring that only the signer can produce a valid signature.
iii. **Verification:** Verification in lattice-based schemes involves checking whether the signature is valid with respect to the public key and a given message.

11.2.3.1 Mathematical Core: SIS-Based Signature

Given:

- Matrix $\mathbf{A} \in \mathbf{Z}_q^{m \times n}$

Find:

- A short non-zero vector $\mathbf{z} \in \mathbf{Z}_n$ such that:

$$\mathbf{Az} \equiv \mathbf{0} \bmod \mathbf{q}$$

This is computationally hard, especially with small $\mathbf{z}$, and forms the foundation for secure signatures.

11.2.3.2 DSS Using SIS

1. Key Generation
Select:
- $\mathbf{A} \in \mathbf{Z}_q^{m \times n}$ (public)
- Trapdoor matrix **T** (private, enables short preimages)

- **Public key: A**
- **Private key: T**

2. Signing a message m
Hash the message:

$$\mathbf{c} = \mathbf{H}(\mathbf{m}) \in \mathbf{Z}_q^m$$

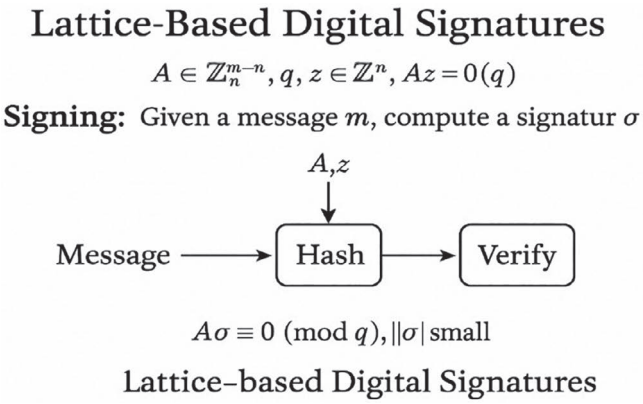


Figure 11.6 Lattice-based digital signature.

Using trapdoor **T**, find a short vector $\mathbf{z} \in \mathbf{Z_n}$ such that:

$$\mathbf{Az} \equiv \mathbf{c} \bmod \mathbf{q}$$

- Signature is $\mathbf{z}$.

3. Verification

Check:

- $\mathbf{Az} \equiv \mathbf{H(m)} \bmod \mathbf{q}$
- $\| \mathbf{z} \|$ is small (within a bound)

☑ If both hold → signature is valid.

11.2.3.2.1 Computational Problems

SVP. Let $B \in Z^{m \times n}$ be a basis for a given lattice L, finding a lattice vector which is non-zero, say, **Bx** such that $\|\mathbf{Bx}\| \leq \|\mathbf{By}\|$ for any other $\mathbf{y} \in Z^n \setminus \{0\}$.

CVP. "Given a lattice basis $B \in Z^{m \times n}$ and a target vector $t \in Z^m$, find the $x \in Z^n$ such that $\|\mathbf{Bx} - \mathbf{t}\|$ is minimum" (Liu et al. 2024).

Approximate SVPγ. Given a lattice basis $B \in Z^{m \times n}$, find a non-zero lattice vector **Bx** such that $\|\mathbf{Bx}\| \leq \gamma \cdot \|\mathbf{By}\|$ for any other $\mathbf{y} \in Z^n \setminus \{0\}$.

Approximate CVPγ. Let $B \in Z^{m \times n}$, be a basis for a given lattice L and a target vector $t \in Z^m$, finding $x \in Z^n$ such that $\|\mathbf{Bx} - \mathbf{t}\| \leq \gamma \cdot \|\mathbf{By} - \mathbf{t}\|$ for any other $\mathbf{y} \in Z^n$. Apart from the precise and estimated compositions, it is likewise possible to articulate such glitches as potentials called **GapSVPγ** and **GapCVPγ.**

11.2.3.2.2 Average-Case Lattice Problems

This segment consists of SIS and LWE.

"SIS. Let **p**, **q**, and **r** be positive integers and β be a real number (positive) less than **r**. $A \in Z_r^{q \times p}$ is an arbitrarily produced matrix ensuing a uniform distribution and designed by **p** independent vectors $ai \leftarrow Z_r$. The problem $\mathbf{SIS}_{q,r,\beta,p}$ enquires to find an SIS $\mathbf{z} \in Z_p$ such that:

$$\mathbf{Az} \equiv \mathbf{0}(\mathbf{mod\, r}), \mathbf{z} \neq \mathbf{0}, \|\mathbf{z}\| \leq \beta\text{."}$$ (Liu et al. 2024)

"**LWE distribution**. Let **m** and **n** be two positive integers, and let a distribution over **Z** be represented by χ. For a fixed secret $s \in Z_n^m$, the LWE distribution $A_{s,Ç} = (x, y)$ over $Z_n^m \times Z_n$ is defined with $y = (x, s) + \mathbf{emodn}$, where $x \leftarrow Z_n^m$, and the error (or noise) $e \leftarrow \chi$ " (Liu et al. 2024).

Search LWE and decisional LWE problems are the two variations of the LWE problems. The first searches the secret **s** with high probability; on the other hand, the second variation differentiates between the uniform distribution and the LWE distribution.

"**Search LWE**. Given **r** independent samples $(x_i, y_i) \in Z_n^m \times Zn$ from $A_{s,\chi}$ and $ei \leftarrow \chi$ for $1 \le i \le r$. The problem $S - LWE_{m,n,\chi,r}$ asks to acquire the secret $s \in Z_n^m$ with high probability ($p > 1 - \delta$)" (Liu et al. 2024).

"**Decisional LWE**. Given $x \in Z_n^m$ **q** and $x \leftarrow Z_n^m$, $s \in Z_n^m$ and $e \in Z_n$ follows the distribution χ. The problem $D - LWE_{m,n,\chi,r}$ asks to distinguish between $(x, s) + e$ and uniform distribution with non-negligible probability" (Liu et al. 2024).

"**Hash function**

A Hash Function is a duo of probabilistic polynomial-time algorithms (Gen, **H**) such that (Gordon et al. 2010):

- Gen is a probabilistic algorithm that on input 1^k outputs a key **s**.
- There exists a polynomial **l** such that **H** takes as input a key **s** and $x \in \{0,1\}^*$ and outputs a string $H_s(x) \in \{0,1\}l(k)$ " (Liu et al. 2024).

The performance comparison of the lattice-based signature schemes is illustrated in Figure 11.7.

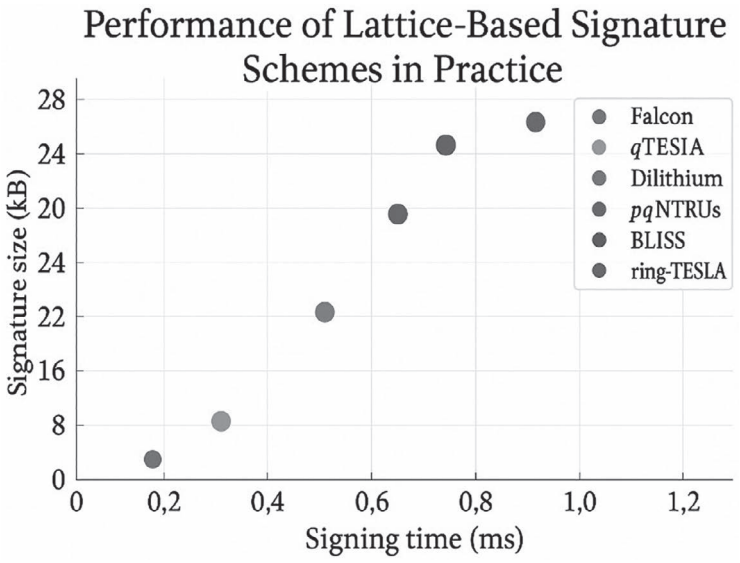


Figure 11.7 Performance analysis graph for the lattice-based signature schemes in practice.

11.3 STATE-OF-THE-ART LATTICE-BASED SIGNATURE

11.3.1 Bimodal Lattice Signature Scheme

The BLISS is a post quantum digital signature scheme based on lattices, specifically the Short Integer Solution (SIS) problem and Ring Learning with Errors (RLWE) problem. It improves upon earlier lattice-based signature schemes by using bimodal Gaussian sampling, which enhances security, efficiency, and compactness. *(Lyubashevsky 2009; Ducas et al. 2013)*

1. Parameters and Notation
   - **q**: Large prime modulus.
   - **n**: Dimension of the polynomial ring (usually a power of 2).
   - $\mathbf{R_q = Z_q[X]/(X^n+1)}$**:** Ring of polynomials modulo **q**.
   - **σ:** Standard deviation for Gaussian sampling.
   - $\mathbf{s_1, s_2}$**:** Secret key polynomials with small coefficients.
   - **A:** Public matrix/polynomial.
   - $\mathbf{t = A\ s_1 + s_2 \bmod q}$**:** Public-key polynomial.
   - **c:** Challenge polynomial.
   - **H:** Cryptographic hash function.
2. Key Generation
   Steps:
   - **Sample secret key polynomials** $\mathbf{s_1, s_2}$**:**
     - Select small polynomials $\mathbf{s_1, s_2 \in R}$ with coefficients in **{−1,0,1}**.
     - **Compute the public-key polynomial:**

       $$\mathbf{t = A\ s_1 + s_2 \bmod q}$$

   - **Output keys:**
     - **Public key:** $\mathbf{pk = (t_1, A)}$.
     - **Secret key:** $\mathbf{sk = (s_1, s_2)}$.
3. Signing Algorithm
   Steps:
   - **Hash the message:**

     $$\mathbf{c = H(M) \in R_q}$$

   - Sample a random polynomial **y** from a bimodal Gaussian distribution $\mathbf{D_\sigma}$.
   - **Compute the masked commitment:**

     $$\mathbf{w = Ay \bmod q}$$

   - **Generate the challenge polynomial:**

     $$\mathbf{c = H(w, M)}$$

   - **Compute the response polynomials:**

     $$\mathbf{z_1 = y + cs_1, z_2 = cs_2}$$

- **Apply rejection sampling:**
  - Accept the signature if $(\mathbf{z_1}, \mathbf{z_2})$ satisfies a norm bound $||\mathbf{z_1}, \mathbf{z_2}|| < \mathbf{B}$.
  - Otherwise, restart the signing process.
- **Output the signature:**

  $\boldsymbol{\sigma = (z_1, z_2, c)}$

4. Verification Algorithm
   Steps:
   1. **Compute commitment reconstruction:**

      $\mathbf{w' = Az_1 - ct \bmod q}$

   2. **Compute challenge polynomial:**

      $\mathbf{c' = H(w', M)}$

   3. **Check validity:**
      - **Accept** if $\mathbf{c = c'}$ and $||\mathbf{z_1}, \mathbf{z_2}||$ is within bound.
      - Otherwise, **reject.**
5. Security and Efficiency
   Security:
   - Based on the NTRU lattice problem and SIS assumption.
   - Bimodal Gaussian sampling ensures that signatures leak minimal information about the secret key.

   Efficiency:
   - Uses fast polynomial arithmetic (FFT-based fast multiplication).
   - Smaller signatures compared to earlier lattice-based schemes.

The BLISS signature scheme is a secure, efficient, and quantum-resistant signature algorithm with strong theoretical foundations.

11.3.2 CRYSTALS-Dilithium Signature Scheme

"The Dilithium signature scheme is a post-quantum digital signature algorithm based on lattice-based cryptography, specifically the CRYSTALS-Dilithium scheme, which is part of NIST's Post-Quantum Cryptography Standardization process" (Ducas et al. 2018).

Dilithium is based on the Fiat-Shamir with Aborts paradigm over structured lattices and provides strong security guarantees under the LWE and SIS assumptions.

The following is a step-by-step explanation of the Dilithium Signature Scheme, including Key Generation, Signing, and Verification, with mathematical details.

1. Parameters and Notation
   - $\mathbf{q}$ be a prime modulus.
   - $\mathbf{n}$ be the lattice dimension.
   - $\mathbf{k,l,d}$ be integer parameters.
   - $\mathbf{R_q = Z_q[X]/(X^n+1)}$ be the polynomial ring.
   - $\mathbf{H}$ be a cryptographic hash function.
   - $\mathbf{U}$ be a uniform sampling algorithm.
   - $\mathbf{D_\eta}$ be a discrete Gaussian distribution with parameter $\boldsymbol{\eta}$.
   - $\mathbf{A} \in \mathbf{Z_q^{k \times l}}$ be a uniformly random matrix.
   - γ_1, γ_2 be bounds controlling noise growth.

The secret keys and public keys are elements in **structured lattices.**

2. Key Generation (Ducas et al. 2018)
   - **Sample secret vectors:**
     - Sample $\mathbf{s_1} \in \mathbf{R_q^l}$, $\mathbf{s_2} \in \mathbf{R_q^k}$ from a **discrete Gaussian distribution** $\mathbf{D_\eta}$ (short-norm polynomials).
     - These are kept as part of the secret key.
   - **Generate the public matrix:**
     - Use a seed ρ to derive a uniformly random matrix $\mathbf{A} \in \mathbf{Z_q^{k \times l}}$.
   - **Compute public-key commitment:**

     $$\mathbf{t = As_1 + s_2 \bmod q}$$

   - **Apply modulus reduction to public key:**
     - Compute a scaled version $\mathbf{t_1}$ by reducing $\mathbf{t}$ modulo $\mathbf{2^d}$:

       $$\mathbf{t_1 = \lfloor t/2^d \rfloor}$$

   - **Output:**
     - **Public key: pk = ($\mathbf{t_1}$, A).**
     - **Secret key: sk = ($\mathbf{s_1}$, $\mathbf{s_2}$, t, ρ).**
3. Signing Algorithm (Bos 2021)
   - **Hash the message and generate randomness:**
     - Compute message hash:

       $$\mu = H(t_1, M)$$

     - Expand a seed ρ to generate a **random masking vector** γ using deterministic sampling.
   - **Sample a short vector y:**
     - Sample $\mathbf{y} \in \mathbf{R_q^l}$ from $\mathbf{D_{\gamma 1}}$ (a bounded distribution).
   - **Compute the commitment w:**

     $$\mathbf{w = Ay \bmod q}$$

   - **Compute the challenge c:**
     - Compute the challenge polynomial $\mathbf{c}$ using a cryptographic hash function:

       $$\mathbf{c = H(w, \mu)}$$

     - This challenge ensures **non-repudiation** by linking the signature to the message.
   - **Compute response z:**
     - Compute:

       $$\mathbf{z = y + cs_1}$$

   - **Check norm bounds:**
     - If $\|\mathbf{z}\|_\infty > {}^{3}\!_{2}$, reject and restart.
   - **Compute hint h:**
     - Compute a hint $\mathbf{h}$ to help reconstruct $\mathbf{w}$ during verification.
   - **Output the signature:**
     - The signature is:

       $$\sigma = (\mathbf{z, c, h})$$

4. Verification Algorithm (Lyubashevsky 2009)
   - **Check norm bound on z:**
     - If $\|\mathbf{z}\|_\infty > {}^{3}{}_{2}$, **reject.**
   - **Recompute the commitment w′:**

     $$\mathbf{w'} = \mathbf{Az} - \mathbf{ct_1 2^d} \bmod \mathbf{q}$$

   - **Recompute the challenge c′:**
     - Compute:

       $$\mathbf{c'} = \mathbf{H(w, \mu)}$$

     - This ensures the challenge was generated correctly.
   - **Check validity:**
     - If **c′=c, accept** the signature. Otherwise, **reject.**
5. Security and Efficiency (NIST PQC Standardization Process)
   Security:
   - Relies on the **Module-LWE and Module-SIS** hardness assumptions.
   - Secure against quantum and classical attacks.
   - Uses **rejection sampling** to ensure signatures leak no information about the private key.

   Efficiency:
   - Optimized polynomial operations use the Number Theoretic Transform (NTT).
   - Compressed signatures and keys reduce bandwidth and storage requirements.
   - Deterministic signing avoids side-channel attacks.

Dilithium provides strong security against quantum attacks and is efficient due to structured lattice computations. It has been selected by NIST as a standard for post-quantum digital signatures.

11.3.3 Falcon Signature Scheme

"The Falcon (Fast Fourier Lattice-based Compact Signatures over NTRU) signature scheme is a lattice-based post-quantum signature algorithm. It relies on the GPV (Gentry-Peikert-Vaikuntanathan) framework for lattice-based signatures and uses the NTRU lattice structure for efficiency and compactness" (Fouque et al. 2018).

Security:

- Based on the hardness of the SIS and NTRU problem.
- Provides provable security under worst-case lattice assumptions.

1. Parameters and Notation (NIST PQC Standardization Process; Gentry et al. 2008)
   - $\mathbf{q}$ be a prime modulus (e.g., $\mathbf{q} = 12289$).
   - $\mathbf{N}$ be the degree of the polynomial ring (usually a power of 2, e.g., $\mathbf{n} = 512$).
   - $\mathbf{R_q = Z_q[X]/(X^n+1)}$ be the ring of polynomials modulo $\mathbf{q}$.
   - $\mathbf{H}$ be a cryptographic hash function.
   - $\mathbf{f, g} \in \mathbf{R}$ be private polynomials.
   - $\mathbf{h = g/f} \bmod \mathbf{q}$ be the public-key polynomial.
   - σ be the Gaussian parameter controlling the signature distribution.

2. Key Generation
   Steps:
   1. **Sample polynomials:**
   - Sample small polynomials $\mathbf{f, g} \in \mathbf{R}$ with small coefficients.
   - Ensure **f** is invertible modulo **q** and $\phi = X^n + 1$.
   2. **Compute public-key polynomial:**

      $$\mathbf{h = g/f \bmod q}$$

   3. **Find auxiliary polynomials F, G:**
      - Solve the NTRU equation:

        $$\mathbf{fG—gF = q \bmod \phi}$$

   4. **Output keys:**
      - **Public key: pk = h**
      - **Secret key: sk = (f, g, F, G)**
3. Signing Algorithm (Gentry et al. 2008)
   Steps:
   1. **Hash the message:**
      - Compute:

        $$\mathbf{c = H(M) \in Rq}$$

   2. **Construct target vector:**

      $$\mathbf{t = (c,0) \in Z^n \times Z^n}$$

   3. **Sample preimage via lattice basis:**
      - Use the secret key lattice basis to find a **short vector** $\mathbf{s = (s_1, s_2)}$ satisfying:

        $$\mathbf{s_1 h + s_2 = c \bmod q}$$

   4. **Use Gaussian sampling:**
      - Apply a discrete Gaussian sampler to ensure **s** is short and statistically independent.
   5. **Output signature:**
      - Signature: $\boldsymbol{\sigma = s_2}$
4. Verification Algorithm
   Steps:
   1. **Hash the message:**

      $$\mathbf{c = H(M)}$$

   2. **Compute linear combination:**

      $$\mathbf{w = \sigma + ch \bmod q}$$

   3. **Check signature norm:**
      - Verify that $||\sigma||$ is within an acceptable bound to prevent forgery.
   4. **Accept or reject:**
      - Accept if **w** satisfies the norm bound.
      - Otherwise, reject.
5. Security and Efficiency
   Security:
   - Based on the NTRU lattice problem and the SIS assumption.
   - Resilient against quantum attacks.

Efficiency:

- Fast key generation and signature operations using Fast Fourier Transform (FFT) techniques.
- **Compact signatures:** Only a few hundred bytes, even at high security levels.

Falcon achieves high security, compact signatures, and fast operations, making it suitable for post-quantum-secure applications.

11.3.4 NTRU Signature Scheme

(NTRU—Nth-degree Truncated Polynomial Ring Unit)

"The NTRU signature scheme is a lattice-based digital signature that leverages the NTRU lattice for security and efficiency. It relies on the hardness of the Short Integer Solution (SIS) and the NTRU problem, which remain difficult even for quantum computers" (Hoffstein et al. 1998; Prest 2018).

1. Parameters and Notation
   - **N**: Degree of the polynomial ring (typically a power of 2).
   - **q:** Large modulus.
   - **p:** Small modulus (often p=3).
   - $\mathbf{R_q = Z_q[X]/(X^n+1)}$: Polynomial ring modulo **q**.
   - $\mathbf{f, g \in R_q}$**:** Private polynomials.
   - $\mathbf{h = f^{-1}\ g \bmod q}$**:** Public-key polynomial.
   - **H:** Cryptographic hash function.
2. Key Generation (Hoffstein et al. 1998)
   Steps:
   1. **Sample polynomials:**
      - Select small polynomials $\mathbf{f, g \in R_q}$ with coefficients in {–1,0,1}.
      - Ensure **f** is invertible modulo **p** and **q**.
   2. **Compute inverses:**
      - Find $\mathbf{f_p^{-1} mod\ p}$ and $\mathbf{f_q^{-1} mod\ q}$.
   3. **Compute public key:**

      $$\mathbf{h = f_q^{-1} g \bmod q}$$

   4. **Output:**
      - **Public key: pk = h**
      - **Secret key:** $\mathbf{sk = \left(f, f_p^{-1}\right)}$
3. Signing Algorithm (Prest 2018)
   Steps:
   1. **Hash the message:**

      $$\mathbf{c = H(M) \in R_p}$$

   2. **Compute initial polynomial:**
      - Randomly select a small polynomial $\mathbf{r \in R_q}$.
   3. **Compute commitment polynomial:**

      $$\mathbf{s = rh + c \bmod q}$$

   4. **Compute the signature polynomial:**

      $$\mathbf{\sigma = fs \bmod q}$$

5. **Reduce modulo p:**

   $\sigma_p = \sigma \bmod p$

6. **Check bounds:**
   - If $||\sigma p||$ is too large, restart.
7. **Output signature:**

   $\sigma = \sigma_p$

4. Verification Algorithm (Hoffstein et al. 1998)
   Steps:
   1. **Hash the message:**

      $c = H(M)$

   2. **Compute verifier polynomial:**

      $v = h\sigma \bmod q$

   3. **Reduce modulo p:**

      $v_p = v \bmod p$

   4. **Check validity:**
      - Accept if $v_p = c$.
      - Otherwise, reject.
5. Security and Efficiency
   Security:
   - Based on the hardness of the NTRU lattice problem and the SIS assumption.
   - Resistant to quantum attacks.

   Efficiency:
   - Fast signing and verification using polynomial arithmetic.
   - Compact keys and small signatures.

The NTRU signature scheme is efficient, compact, and secure against both classical and quantum attacks.

11.3.5 qTESLA Signature Scheme

qTESLA is a lattice-based digital signature scheme designed for post-quantum security. It is based on the Ring Learning With Errors (RLWE) problem, ensuring security even against quantum attacks. The scheme was submitted to the NIST PQC competition, and while it was not selected for standardization, it remains an important example of an efficient lattice-based signature scheme. *(Bindel et al. 2018; Lyubashevsky 2009)*

1. Parameters and Notation
   - **n:** Dimension of the polynomial ring.
   - **q:** Large prime modulus.
   - **p:** Small modulus for coefficient reduction.
   - **σ:** Standard deviation for Gaussian sampling.
   - $R_q = Z_q[X]/(X^n+1)$: Ring of polynomials modulo **q**.
   - **s, e:** Small secret key polynomials.
   - **A:** Public-key polynomial/matrix.
   - **t:** Public-key polynomial.

- **c:** Challenge polynomial.
- **H:** Cryptographic hash function.

2. Key Generation
   Steps:
   - **Sample small secret key polynomials s, e:**
     - Sample $\mathbf{s}, \mathbf{e} \in \mathbf{R}_q$ from a **discrete Gaussian distribution** $\mathbf{D}_\sigma$.
   - **Compute the public-key polynomial:**

     $$\mathbf{t = As + e \bmod q}$$

   Output keys:
   - **Public key: pk = (A, t).**
   - **Secret key: sk = (s, e).**
3. Signing Algorithm
   Steps:
   - **Hash the message:**

     $$\mathbf{c = H(M)} \in \mathbf{R}_q$$

     - Sample a random polynomial **y** from a Gaussian distribution $\mathbf{D}_\sigma$.
     - **Compute the commitment:**

       $$\mathbf{w = Ay \bmod q}$$

     - **Generate the challenge polynomial:**

       $$\mathbf{c = H(w, M)}$$

     - **Compute the response polynomial:**

       $$\mathbf{z = y + cs}$$

     - **Apply rejection sampling:**
       - Accept the signature if ||**z**|| is within a specified bound.
       - Otherwise, **restart the signing process.**
     - **Output the signature:**

       $$\boldsymbol{\sigma} = (\mathbf{z}, \mathbf{c})$$

4. Verification Algorithm
   Steps:

   - **Recompute the commitment:**

     $$\mathbf{w' = Az—ct \bmod q}$$

   - **Recompute the challenge polynomial:**

     $$\mathbf{c' = H(w', M)}$$

   - **Check validity:**
     - **Accept** if $\mathbf{c = c'}$ and ||**z**|| is within bound.
     - Otherwise, **reject.**
5. Security and Efficiency
   Security
   - Based on the **RLWE problem.**
   - Uses **Gaussian sampling** to prevent **side-channel attacks.**
   - Resistant to **quantum attacks.**

Efficiency:
- Uses **fast polynomial arithmetic** (NTT for efficient multiplication).
- **Smaller key and signature sizes** compared to other lattice-based schemes.

The qTESLA signature scheme is a secure, efficient, and quantum-resistant lattice-based signature algorithm designed for PQC.

Traditional lattice-based schemes for the erections of lattice-based digital signature are summarized in Table 11.1 (Liu et al. 2024) also, Figure 11.8 shows signature sizes of different cryptographic schemes and Figures 11.9 and 11.10 show Security levels of different cryptographic schemes:

11.4 SPECIFIC DIGITAL SIGNATURES

11.4.1 Group Signatures and Ring Signatures

The group signature is a certain scheme primarily proposed by Chaum and Van Heyst as a digital signature in 1991 (Chaum and Van Heyst 1991). In group signature, each group

Table 11.1 Conventional Schemes

Schemes	*Signature size (kb)*	*Security (bits)*
GLP	21.5	80
BLISS I	12.6	128
BLISS II	12	128
BLISS III	13	160
BLISS IV	13.5	192
ring-TESLA I	36.5	80
ring-TESLA II	36	128
Dilithium	15	128

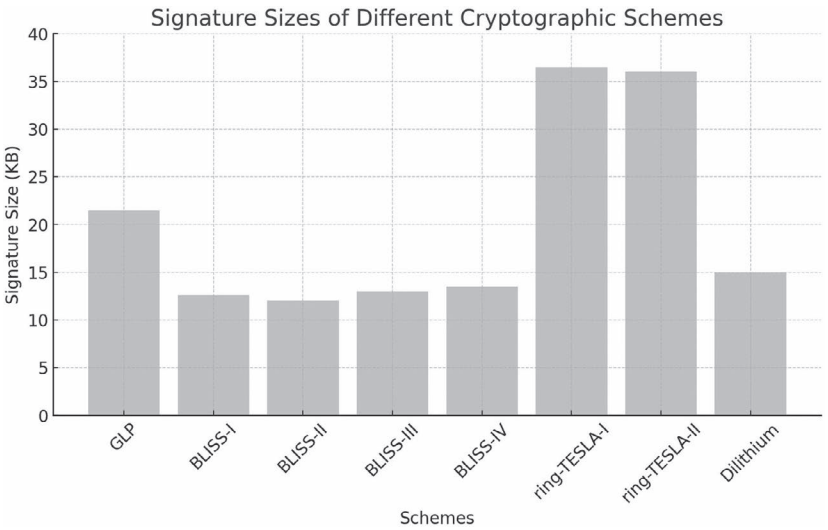


Figure 11.8 Signature sizes of different cryptographic schemes.

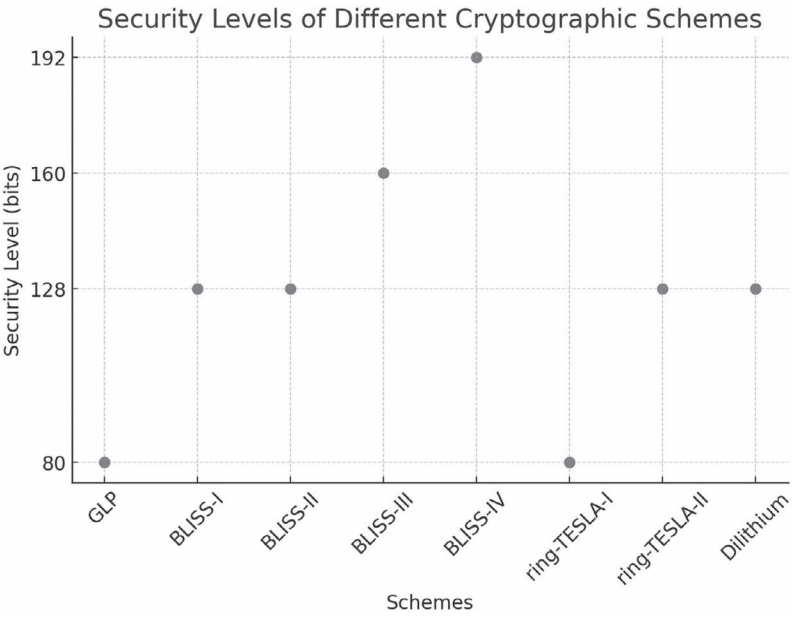


Figure 11.9 Security levels of different cryptographic schemes.

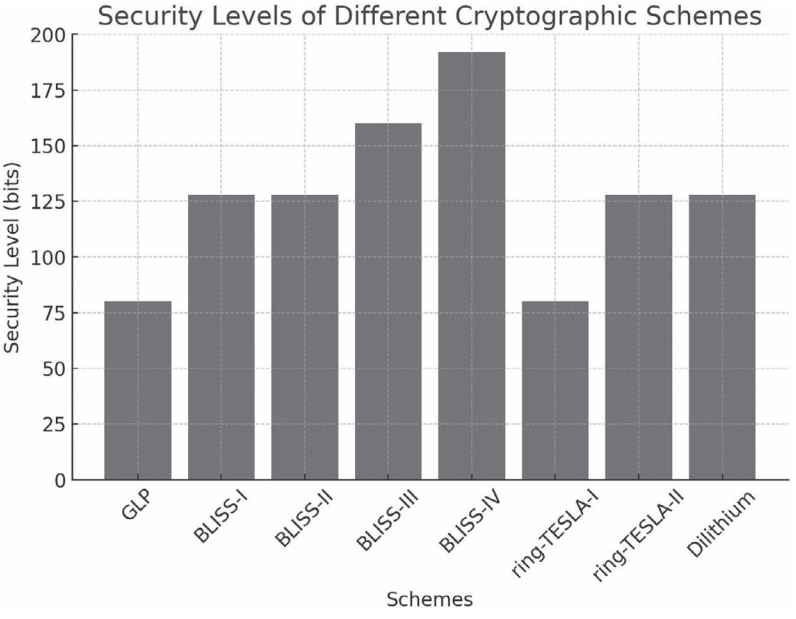


Figure 11.10 Security levels of different cryptographic schemes.

member is able to sign for other members or for the whole group, which is in an unidentified way. Group signatures preserve two vital possessions: concealment and traceability. The schemes' security stated in Chaum and Van Heyst (Chaum and Van Heyst 1991) is constructed with the complexity of factorization, and, for large integers, discrete logarithm problems, which are assumed to be slightly feeble in the post-quantum era.

In 1994, Chen and Pedersen introduced a group signature scheme—referred to as partially dynamic group signatures—that allowed novel associates to be added dynamically after the initial

arrangement (Chen and Pedersen 1994). However, in these schemes, the size of both the signatures and the public keys increases in direct proportion to the total count of group members, which is highly problematic as far as large groups concerned; to overcome this limitation, a scheme was proposed in 1997 known as CS97 group signature decoupling the public key and signature sizes from the group size. Later, in 2000, the ACJT scheme was presented as a group signature scheme (Camenisch and Stadler 1997; Ateniese et al. 2000) based on classical number theoretic problems. Following these developments, Kim et al. (2001) were the first to propose a scheme that permitted members either to exit voluntarily or to be removed by administrators, and, subsequently, a BMW model was launched. Bellare et al. (2003) provided a conceptual framework for static group signatures. The first lattice-based group signature scheme was presented by (Gordon et al. 2010), which demonstrated the amalgamation of lattice theory and group signatures. Bootle et al. (2016) introduced a go-ahead signature scheme along with firm security definitions. (Libert et al. 2016) proposed a scheme that included a mechanism for adding new members. However, the enrollment process was extremely intricate and time-consuming, and the scheme did not support member revocation. Ling et al. (2017) developed a fully dynamic lattice-based group signature scheme in 2017 that incorporated both member addition and cancellation mechanisms, utilizing a Merkle hash tree.

Laguillaumie et al. (2013) introduced improvements to the signature size, limiting its dependence on the number of members of the group to terms which are logarithmic. Luo and Jiang (2020) proposed an RSIS and RLWE hard problem-based scheme, ensuring properties such as secrecy and traceability. (Ling et al., 2017) developed a lattice-based scheme utilizing RSIS and RLWE, which was a partially dynamic signature scheme. Additionally, Xie et al. (2019) introduced an SIS- and LWE-based scheme, achieving full secrecy, traceability, and non-frameability. Building on this, Sun et al. (2019) further refined a scheme based on RLWE and RSIS.

(Tang et al., 2021) proposed a ring signature scheme, which was an identity-based, linkable ring signature scheme that leveraged hatches and an accept–reject algorithm, effectively plummeting time (Tang et al. 2021). Baum et al. (2018) introduced a scheme constructed on the lattice-based RSIS hard problem. Ring signatures have diverse applications, including anonymous voting, medical data sharing, and vehicular networks. However, lattice-based ring signatures that can withstand quantum attacks are still in the developmental stage, and designing quantum-resistant ring signature schemes remains a significant challenge.

The key sizes, signature size, and their relationship with N are summarized in Table 11.2.

Table 11.2 The Key Sizes (Group Public Key and Signing Key), Signature Size, and Relationship for Lattice-Based Digital Signature Schemes between Signature and N (Liu et al. 2024)

Schemes	*gpk size*	*sk size*	*sig size*	*Relationship*
Gordon et al. (2010)	$O(\lambda^2 N)$	$O(\lambda^2)$	$O(\lambda^2 N)$	Linear
Laguillaumie et al. (2013)	$O(\lambda^2 \log N)$	$O(\lambda^2)$	$O(\lambda \log N)$	Logarithmic
Libert et al. (2016)	$O(\lambda^2 \log N)$	$O(\lambda)$	$O(\lambda \log N)$	Logarithmic
Ling et al. (2017)	$O(\lambda)$	$O(\lambda)$	$O(\lambda)$	Constant
Xie et al. (2019)	$O(N)$	$O(\lambda)$	$O(N)$	Linear
Sun et al. (2019)	$O(\lambda + \log\lambda \log N)$	$O(\lambda + \log\lambda \log N)$	$O(\lambda + \log\lambda \log N)$	Logarithmic
Luo and Jiang (2020)	$O(\lambda \log^2 N)$	$O(\lambda \log^2 N)$	$O(\lambda \log^3 N)$	Constant

11.4.2 Blind Signatures

With the aim of enhancing automatic payment systems' security, Chaum (1983) introduced a novel cryptographic notion, the blind signature. The scheme involved a user and a signer interface process. The user first does a blind alteration to facade the plan text and then transmits the malformed text to the signer using the public key. In the final step, the user applies a converse transformation to attain the sign conforming to the plan text. This scheme guarantees that the signer remains unaware of which texts have been signed and is unable to trace which signature resulted from a specific signing process.

Due to its properties of blindness and unforgeability, the blind signature is extensively applied in areas like e-voting (Shao 2003; Cruz and Kaji 2017), e-cash (Li et al. 2017; Aboud and Al-Fayoumi 2007), among others, where ensuring the privacy of the message sender is essential.

In 1995, Camenisch et al. (1995) introduced a discrete logarithm problem-based blind signature scheme. Two innovative schemes later emerged—first derived from a modified version of DSA (Wang and Hou 2019) and second built upon Nyberg-Rueppel's signature scheme (Ling et al. 2017). However, since this scheme was introduced at an early stage, its precision and proficiency were not evaluated in the paper. Following this, numerous blind signature schemes based on the discrete logarithm problem and the integer factorization problem were developed. For instance, Harn (1995) proposed another blind signature scheme in 1995 relying on the discrete logarithm problem and introduced the concept of strong blind signatures, which sparked further research and advancements in this field. Additionally, in 1996, Pointcheval and Stern (1996) established the formal security definition of blind signatures, outlining that their security is characterized by two key properties: blindness and one more unforgeability.

As the advent and advancement of quantum computers is nearing, the safety of blind signature schemes based on conventional number theoretic problems has been greatly weakened, posing significant challenges to this domain. However, lattice-based cryptosystems offer various advantages among post-quantum cryptosystems.

Rückert (2010) introduced the first lattice-based blind signature scheme, incorporating Lyubashevsky's filtering technique (Lyubashevsky 2009). Wang et al. (2010), by utilizing a preimage sampling function, enhanced the blind signature algorithm, enabling the scheme to be executed in just two rounds of interaction. This optimization outperformed Rückert's scheme (Rückert 2010) and ensured both blindness and unforgeability. Gu et al. (2012) developed an identity-based signature scheme based on lattices and introduced its blind signature variant. This form guaranteed unforgeability and blindness within the arbitrary oracle model while producing shorter private keys and signatures. Gao et al. (2017) anticipated two identity-based blind signature schemes from lattices, one constructed within the arbitrary oracle model and the other within the usual model. Tang et al. (2017) presented an identity-based blind signature scheme in the usual model. This scheme employed a basis delegation algorithm to generate private keys based on user identities, while a forward sampling algorithm was used for signing messages. The scheme satisfied one more unforgeability, with its security relying on the SIS problem. Zhu et al. (2018) introduced an identity-based blind signature scheme using the NTRU lattice (Liu et al. 2024). Instead of constructing a trapdoor, this scheme primarily utilized a reject sampling theorem to ensure security within the random oracle model, offering benefits such as confidentiality, integrity, and non-repudiation. However, (Singh and Padhye, 2020) identified a security susceptibility in this scheme and provided an improved version. Li et al. (2021) introduced a lattice-based blind signature scheme designed for blockchain systems. This scheme leveraged a bimodal Gaussian distribution and rejection of sampling for

signing, achieving both blindness and one-more unforgeability in the arbitrary oracle model while enhancing the probability of efficacious signing.

Lyubashevsky et al. (2022) introduced a two-round optimal lattice-based blind signature scheme. The scheme looks like the most effective blind signature candidate at present.

As blind signature technology continues to advance, its extended ideas and integrated methodical approaches have been extensively adopted. Lin and Jan (2000) introduced the concept of proxy blind signature by merging proxy signature with blind signature.

The signature size, security model, and number of moves for the scheme are summarized in Table 11.3.

11.4.3 Proxy Signatures

The idea of proxy signature was primarily introduced by (Mambo et al. 1996). The impetus of the proxy signature proposal is to execute protected allocation of signature dominion, that is, by presenting a proxy signer that can sign on behalf of the actual signer, and the proxy cannot falsify the signature of the actual signer. Based on the level of signing authorization, proxy signatures can be categorized into fully authorized (Kim et al. 2013), partial proxy, and proxy signatures with certificates. In the fully authorized model, the original signer hands over the secret key used for signing directly to the proxy signer, who then uses it to sign messages. However, because there is no distinction between the original and proxy signatures, this scheme fails to meet the requirement of non-repudiation.

Shum and Wei (2002) introduced a proxy signature scheme that relies on the discrete logarithm problem, where the proxy signer's identity was masked using an alias, and only a designated authority had the ability to disclose their true identity. Furthermore, several proxy signature schemes had been developed based on classical mathematical problems. These include schemes utilizing the discrete logarithm problem, as proposed by Li et al. (2003) and Hwang and Chen (2003), as well as schemes built upon the integer factorization problem, such as the one introduced by (Shao 2003), all of which were published in 2003.

In the age of post-quantum, the research emphasis has focused on quantum-secure proxy signature schemes, as conventional public-key cryptosystems are now vulnerable. Jiang et al. (2010) proposed a lattice-based proxy signature scheme with the bonsai tree model (Cash et al. 2012). It is constructed upon the GPV signature scheme by Gentry et al. (2008), which depends on a set of preimage-sampleable hatch functions. Nevertheless, a key drawback of this scheme is that it fails to provide protection for the proxy signer, thereby enabling the original signer to forge signatures on behalf of the proxy. In reply, Xia et al. (2011) introduced a lattice-based proxy signature scheme that leveraged hatch functions with preimage sampling and the bonsai tree model. The security of this scheme is rooted in the hardness of the average-case SIS and inhomogeneous SIS problems. Although the public and secret keys

Table 11.3 Signature Size, Security Model, and Move for Scheme (Liu et al. 2024)

Schemes	*Sig size*	*Security model*	*Move*	*Unforgeability*
"Rückert (2010) (n = 2048)"	$"(n+m)\log q"$	"Random Oracle"	4	"yes"
" Wang et al. (2010)**"**	$"m\log q"$	"Random Oracle"	**2**	"yes"
" Gao et al. (2017)**"**	$"m\log q"$	"Standard Model"	**2**	"yes"
" Tang et al. (2017)**"**	$"m\log q"$	"Standard Model"	**2**	"yes"

in this approach are greater than those in schemes based on factoring or discrete logarithm problems, it only necessitates linear operations on small integers. To tackle the matter of fluctuating key sizes in proxy signature schemes based on the bonsai tree principle, Yu (2013) proposed a scheme in 2013 that allowed for manageable signature length. This scheme utilized a fixed-dimension lattice-based delegation algorithm to derive the proxy key and incorporated a preimage sampling function to parameterize the proxy signature. Its security was established on the trouble of solving the SIS problem and the SVP in lattices. Kim et al. (2013) developed an identity-based proxy signature scheme founded on lattice problems, utilizing a fixed-dimensional lattice-based delegation technique. Notably, this was the first scheme to offer protection for the proxy under the adaptive security model. Li et al. (2003) introduced a lattice-based proxy signature scheme that was provably secure within the standard model. This scheme primarily relied on the preimage sampling algorithm, ensuring existential unforgeability against compliant preferred message attacks, built on the SIS problem in the standard model. Jiang et al. (2014) designed a proxy signature scheme incorporating trapdoor-free signatures and small-norm matrix transfer techniques, with security based on the SIS problem. While this approach reduced the size of both the secret key and the proxy signature, it lacked proof of public verifiability for proxy authorization. To address this, Lu et al. (2016) introduced the concept of authorization certificates in 2016, enhancing the scheme previously proposed by Jiang et al. (2014) and (Liu et al. 2024).

Zhu et al. (2018) introduced an identity-based proxy signature scheme utilizing the NTRU lattice, which was proven secure under the random oracle model. This scheme offers the advantage of generating relatively small signatures and key sizes. Xie et al. (2021) proposed a forward-secure lattice-based proxy signature scheme. As its name suggests, the scheme ensures forward security; however, enhancing its security comes at the expense of efficiency.

The comparison of group, ring, blind, and proxy signature schemes in terms of signature size, group public-key size, and signing key size is presented in Table 11.4.

Recent developments in lattice-based DSSs have introduced several noteworthy proposals:

VDOO: Introduced in December 2023, VDOO is a post-quantum digital signature algorithm based on solving multivariate equations. It enhances security by adding a diagonal layer over the oil-vinegar-based signature scheme Rainbow, resulting in improved performance and reduced signature sizes.

Compact Lattice Gadget Framework: Published in May 2023, this framework aims to improve the practicality of gadget-based cryptosystems, focusing on hash-and-sign signatures. It introduces a compact gadget framework with a specialized sampler, leading to more efficient and secure NTRU and RLWE-based signature schemes.

Table 11.4 Group, Ring, Blind, and Proxy Signature Schemes on the Basis of Signature Size, Group Public-Key Size and Signing Key (Liu et al. 2024)

Signature scheme	*Signature size*	*Group public-key size*	*Signing key size*
Group Signature	Moderate to large	Large (depends on group size)	Small (individual key)
Ring Signature	Large (increases with ring size)	No centralized group key	Small (individual key)
Blind Signature	Moderate (depends on underlying scheme)	Standard (same as normal signature scheme)	Standard (same as normal signing key)
Proxy Signature	Moderate to large (depends on delegation scheme)	Standard or large (if delegation keys are included)—security against misuse	Standard (original + delegated key)

Squirrels: Submitted to the NIST Call for Additional Post-Quantum Signature Schemes, Squirrels is a DSS that trails the hash-and-sign model over unstructured co-cyclic integer lattices. It emphasizes conservative security assumptions and achieves compact signatures, making it a promising candidate for standardization.

These schemes represent the forefront of research in lattice-based digital signatures, each contributing unique approaches to enhance efficacy, privacy, and security in the era of PQC.

11.5 CHALLENGES AND ISSUES IN LATTICE-BASED DIGITAL SIGNATURES

- Choosing appropriate lattice parameters (e.g., dimension, basis, modulus) is crucial without sacrificing efficiency.
- Generating large, secure keys can be computationally intensive due to the need for finding suitable lattice bases.
- Lattice-based signatures tend to be larger compared to some other schemes, impacting transmission and storage costs.
- Reliance on the hardness of lattice problems for security, which could potentially be weakened by advances in quantum computing or algorithmic breakthroughs.
- Signing operations can be slower compared to traditional schemes, due to the complexity of lattice operations.

11.6 CONCLUSION AND FUTURE SCOPE

This chapter reviewed the mathematical concepts used in lattice-based digital signature schemes (DSSs). Also, this survey explores systematic progress of lattice-based DSSs. The analysis starts with conventional lattice-based digital signature schemes, progresses to specialized DSSs, and culminates with the most recent lattice-based DSSs, comparing them in terms of signature size, group public-key size, and signing key size. Also, this survey highlights the challenges and issues in lattice-based digital signatures. However, the schemes are evolving with time for the enhancement of security (Kowsalya et al. 2021), and these lattice-based DSSs are assumed to be quantum attack resistant to some extent, but still we need a more secure and advanced scheme to resist the quantum attacks in the near future.

Despite significant progress, improving the efficiency of lattice-based digital signatures remains an open problem. Ensuring that the lattice parameters chosen for digital signatures remain secure against quantum attacks is crucial. Research is ongoing to develop parameters that are resistant to both classical and quantum attacks, considering potential advances in quantum computing. Developing techniques and protocols that effectively mitigate side-channel resistance without compromising efficiency is an active area of research (Navalakhe and Atre 2021). NIST-standardized signatures, Dilithium and Falcon, are also vulnerable to side-channel attack; therefore, to construct a secure signature in the current world is still an open problem.

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Chapter 12

Construction of Some New Classes of Irreducible and Normal Polynomials over Finite Fields

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12.1 INTRODUCTION

The exploration of irreducible polynomials (IRPs) and normal polynomials (NPs) over finite fields is of significant importance in both theoretical mathematics and practical domains such as coding theory, cryptography, and digital communication systems. IRPs are essential for creating finite field extensions, which serve as a foundation for designing error-correcting codes and secure encryption protocols (Lidl and Niederreiter 1983; Mullen and Panario 2013). These polynomials are fundamental in forming Galois fields that are employed in public-key cryptosystems and pseudorandom number generators. In parallel, NPs facilitate the construction of normal bases, enabling efficient arithmetic operations in finite fields, key for computational uses in coding theory and digital communication. Notably, Reed-Solomon and BCH codes rely on IRPs for effective error correction and detection. Additionally, IRPs are critical in the development of linear feedback shift registers, which are extensively utilized in stream cipher encryption. Iterative methods for constructing IRPs are particularly valuable for generating endless sequences of such polynomials see, (Kyuregyan 2002; Varshamov 1984; Sharma et al. 2022; Sharma et al. 2015; Alizadeh 2017; Menezes et al. 2013; Cheng 2024). The composition method stands out as a robust technique for studying and generating polynomials over finite fields, also aiding in tasks such as root extraction and polynomial factorization (Abrahamyan et al. (2012), Abrahamyan and Kyureghyan (2011); Alizadeh and Mehrabi (2015); Lidl and Niederreiter (1983); Kyuregyan (2002); Kyuregyan (2004). Recursive construction methods, particularly those involving compositions with quadratic rational functions, have been detailed in works such as Kim and Son (2020); Kyuregyan (2008); Panario et al. (2020), and Gao (1993). A field element λ_1 in an extension field is called a normal element if the set $\lambda_1, \lambda_1^q, \ldots, \lambda_1^{q^{n-1}}$ forms a linearly independent basis over $\mathbb{F}_q$. An IRP of degree n over $\mathbb{F}_{q^n}$ with at least one root being a normal element is termed an NP. Kyuregyan (2008) introduced a recursive method to construct NPs using the transformation $x \mapsto \gamma + \lambda^2 x^{-1}$. Author later expanded on this idea using the composition method to generate families of IRPs see, Kyuregyan and Kyureghyan (2010). Kim and Son (2020) proposed a transformation $\frac{x^p - x^{p-1}+1}{-x^{p-1}+1}$ to develop an infinite sequence of IRPs and NPs. Building on these foundations, Sharma et al. (2022) utilized the transformation $x \mapsto \frac{x^p - x^{p-1}+1}{x^p}$ to create similar sequences. More recent advancements include the use of isogeny-based transformations from elliptic curves to generate IRPs see, Bassa et al. (2024). Cherchem et al. (2022) presented a construction method leveraging the Frobenius automorphism and companion matrices to derive irreducible and primitive polynomials, with direct applications in the generation of BCH codes.

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From an algebraic standpoint, this chapter proposes a novel transformation $x \mapsto \frac{x^p - x + 1}{x^p - x}$ to generate IRPs and NPs over the field $\mathbb{F}_q$, where $q = p^n, p$ is a prime and n a positive integer. The primary goal is to extend existing research by introducing new iterative strategies that facilitate efficient construction of such polynomials while preserving computational feasibility. Drawing inspiration from earlier transformations, we develop new families of IRPs and NPs.

Organization of the Chapter

The rest of the chapter is organized as follows. Section 12.2 outlines necessary preliminaries. Section 12.3 presents the construction of IRPs. Section 12.4 provides the method for building NPs from a given initial NP. Each theorem and lemma is supported by concrete examples using Mathematica 9.3 and SageMath 9.3. And, finally, Section 12.5 concludes the chapter.

12.2 PRELIMINARIES

Some definitions and propositions related to the manuscript are given here:

Definition 12.2.1 Mullen and Panario (2013). For $\lambda_1 \epsilon \mathbb{F}_{q^n}$, trace $\mathrm{Tr}_{q^n/q}(\lambda_1)$ of λ_1 over $\mathbb{F}_q$ is defined by

$$\mathrm{Tr}_{q^n/q}(\lambda_1) = \lambda_1 + \lambda_1^q + \lambda_1^{q^2} + \cdots + \lambda_1^{q^{n-1}}.$$

Definition 12.2.2 Lidl and Niederreiter (1983). For an n-degree polynomial $g(x) \epsilon \mathbb{F}_q[x]$, its reciprocal polynomial is

$$g^*(x) = x^n g\left(\frac{1}{x}\right)$$

Proposition 12.1.1 Cohen (1969). Let $s(x), t(x) \in \mathbb{F}_q[x]$ whose greatest common divisor is one and $P(x)$ is an n-degree IRP, then composition

$$H(x) = t^n(x) P\left(\frac{s(x)}{t(x)}\right)$$

over $\mathbb{F}_q$ is irreducible if and only if $s(x) - \beta t(x)$ over $\mathbb{F}_q$ is an IRP, for any root $\beta \in \mathbb{F}_{q^n}$ of $P(x)$.

Proposition 12.1.2 Menezes et al. (2013). The trinomial $x^p - x - b$ is irreducible if and only if $\mathrm{Tr}_{q/p}(b) \neq 0$, where $\beta \in \mathbb{F}_q$.

Proposition 12.1.3 Alizadeh et al. (2018). Let $\lambda_1 \in \mathbb{F}_{q^n}$ such that $\lambda_1 \notin \mathbb{F}_q$ and $\delta \in \mathbb{F}_p^*$. Then,

$$\sum_{t=0}^{p-1} \frac{1}{\gamma_1 + t\delta} = -\frac{1}{\gamma_1^p - \gamma_1} \tag{12.1}$$

Proposition 12.1.4 Menezes et al. (2013). Assume $h(x) \in \mathbb{F}_q[x]$ is an n-degree IRP having a root β and $p_i(x)$ is defined as in Equation (12.2*). Then, the polynomial $h(x)$ is normal if and only if

$$L_{p_i}(\beta) \neq 0, \forall i = 1,2,\ldots,k, \tag{12.2}$$

$$p_i(x) = \frac{x^n - 1}{l_i(x)} = \sum\nolimits_{u=0}^{v_i} c_{iu} x^u, i = 0,1,2,3,\ldots,k \tag{12.2*}$$

for a linearized polynomial $L_{p_i(x)} = \sum\nolimits_{u=0}^{v_i} c_{iu} x^{q^u}$ over $\mathbb{F}_q$.

12.3 CONSTRUCTION OF IRPS

In the current section, IRPs are constructed with the help of the transformation $x \mapsto \frac{x^p - x + 1}{x^p - x}$ over the finite field $\mathbb{F}_q$.

First, a lemma is provided for the irreducibility of the composition of polynomials using the given transformation. Examples of IRPs are given using SageMath and Mathematica.

Theorem 12.3.1 Consider $h(x) = \sum\nolimits_{j=0}^{n} h_j x^j$ over $\mathbb{F}_q$ be an n-degree IRP and $\lambda \in \mathbb{F}_p^*$. Then the polynomial

$$H(x) = \left(x^p - x\right)^n h\left(\frac{x^p - x + 1}{x^p - x}\right)$$

is irreducible over $\mathbb{F}_q$ with degree pn if and only if

$$\mathrm{Tr}_{q/p}\left(\frac{h'(1)}{h(1)}\right) \neq 0 \tag{12.3}$$

Proof. Let an IRP $h(x)$ have root β in $\mathbb{F}_{q^n}$. Then, by Proposition 2.1, irreducibility of $H(x)$ over $\mathbb{F}_q$ implies irreducibility of $x^p - x + 1 - \beta\left(x^p - x\right)$ and vice versa, which is equivalent to the irreducibility of $x^p - x - \left(\frac{1}{\beta - 1}\right)$ over $\mathbb{F}_{q^n}$. Hence, using Proposition 2.2, $x^p - x - \left(\frac{1}{\beta - 1}\right)$ is irreducible over $\mathbb{F}_{q^n}$ if and only if $Tr_{q^n/p}\left(\frac{1}{\beta - 1}\right) \neq 0$. Now,

$$\mathrm{Tr}_{q^n/p}\left(\frac{1}{\beta - 1}\right) = \mathrm{Tr}_{q/p}\left(\mathrm{Tr}_{q^n/q}\left(\frac{1}{\beta - 1}\right)\right),$$

Now,

$$h(x) = h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \cdots + h_n x^n$$

So,

$$h\left(\frac{x+1}{x}\right)=h_0+h_1\left(\frac{x+1}{x}\right)+h_2\left(\frac{x+1}{x}\right)^2+h_3\left(\frac{x+1}{x}\right)^3+\cdots+h_n\left(\frac{x+1}{x}\right)^n.$$

Since,

$$\begin{aligned} Tr_{q^n/p}\left(\frac{1}{\beta-1}\right)&=Tr_{q/p}\left(Tr_{q^n/q}\left(\frac{1}{\beta-1}\right)\right)\\ Tr_{q^n/q}\left(\frac{1}{\beta-1}\right)&=\frac{-h'(1)}{h(1)}\end{aligned} \tag{12.4}$$

So,

$$\mathrm{Tr}_{q^n/p}\left(\frac{1}{\beta-1}\right)=-\mathrm{Tr}_{q/p}\left(\frac{h'(1)}{h(1)}\right)$$

Hence, $H(x)$ is irreducible if and only if

$$\mathrm{Tr}_{q/p}\left(\frac{h'(1)}{h(1)}\right)\neq 0.$$

Example 12.3.1. Consider $g(x)=x^2+2x+2$ be an IRP having root a such that $\mathbb{F}_{3^2}=\mathbb{F}_3(a)$ and

$$h(x)=(a+1)x^2+(2a+1)x+2a \tag{12.5}$$

be an IRP over $\mathbb{F}_{3^2}$.

$$h'(x)=2(a+1)x+(2a+1)$$

This implies,

$$h'(1)=a$$

and

$$h(1)=2a+2$$

Now,

$$\begin{aligned}\mathrm{Tr}_{3^2/3}\left(\frac{h'(1)}{h(1)}\right)&=\mathrm{Tr}_{3^2/3}\left(\frac{a}{2a+2}\right)\\ &=\mathrm{Tr}_{3^2/3}\left(a(a+1)\right)\\ &=(2a+1)+(2a+1)^3=1\neq 0\end{aligned}$$

Since all the conditions of Theorem 3.1 are fulfilled, the polynomial

$$\begin{aligned} H(x) &= \left(x^3-x\right)^2 h\left(\frac{x^3-x+1}{x^3-x}\right) \\ &= \left(x^3-x\right)^2\left((a+1)\left(\frac{x^3-x+1}{x^3-x}\right)^2+(2a+1)\left(\frac{x^3-x+1}{x^3-x}\right)+2a\right) \\ &= (5a+2)x^6-(10a+4)x^4+(4a+3)x^3+(5a+2)x^2-(4a+3)x+a+1 \\ &= (2a+2)x^6+(2a+2)x^4+ax^3+2ax^2+2x^2+2ax+a+1 \end{aligned}$$

Example 12.3.2 Consider $g(x)=x^3+x^2+x+2$ be an IRP having root a such that $\mathbb{F}_{3^3}=\mathbb{F}_3(a)$ and

$$h(x)=ax^3+ax^2+(a+1)x+a \tag{12.6}$$

is an IRP over $\mathbb{F}_{3^3}$. Now, we prove that

$$\mathrm{Tr}_{3^3/3}\left(\frac{h'(1)}{h(1)}\right)\neq 0 \tag{12.7}$$

By taking the derivative of Equation (12.6), we get

$$h'(x)=3ax^2+2ax+(a+1) \tag{12.8}$$

Putting $x=1$ in Equations (12.6) and (12.8), we get

$$h(1)=a+1 \text{ and } h'(1)=1 \tag{12.9}$$

Using Equation (12.9) in Equation (12.7),

$$\begin{aligned} \mathrm{Tr}_{3^3/3}\left(\frac{h'(1)}{h(1)}\right) &= \mathrm{Tr}_{3^3/3}\left(\frac{1}{a+1}\right) \\ &= \mathrm{Tr}_{3^3/3}\left(2a^2+2\right) \\ &= 1\neq 0 \end{aligned}$$

Since all the assumptions of Theorem 3.1 are satisfied, the polynomial

$$\begin{aligned} H(x) &= \left(x^3-x\right)^3 h\left(\frac{x^3-x+1}{x^3-x}\right) \\ &= \left(x^3-x+2\right)^3\left(a\left(\frac{x^3-x+1}{x^3-x}\right)^3+a\left(\frac{x^3-x+1}{x^3-x}\right)^2+(a+1)\left(\frac{x^3-x+1}{x^3-x}\right)+a\right) \\ &= (4a+1)x^9+(-12a-3)x^7+(6a+1)x^6+(12a+3)x^5+(-12a-2)x^4-x^3 \\ &\quad +(6a+1)x^2-4ax+a \\ &= (a+1)x^9+x^6+x^4+2x^3+x^2+2ax+a \end{aligned}$$

over $\mathbb{F}_{3^3}$ is irreducible with degree 9.

Theorem 12.3.2 For any IRP, $H(x)=\sum_{i=0}^{n} h_i x^i$ over $\mathbb{F}_q$ of degree n, the polynomial

$$t(x)=H\left(\frac{x^p-x^{p-1}+1}{x^{p-1}-1}\right)\cdot\left(x^{p-1}-1\right)^n$$

over $\mathbb{F}_q$ is a pn degree IRP if and only if

$$\mathrm{Tr}_{q/\mathrm{p}}\left(\frac{H'(1)}{H(1)}\right)\neq 0$$

Proof. By applying Proposition 2.1, $t(x)$ over $\mathbb{F}_q$ is an IRP if and only if $\left(x^p-x^{p-1}+1\right)-\gamma\left(x^{p-1}-1\right)$ is an IRP over $\mathbb{F}_{q^n}$, where $\gamma\in\mathbb{F}_q$ is a root of $H(x)$. Precisely, $t(x)$ is an IRP if and only if $x^p-x-\left(\frac{-1}{\gamma+1}\right)$ is an IRP over $\mathbb{F}_{q^n}$. Also, using Proposition 2.2. $x^p-x-\left(\frac{-1}{\gamma+1}\right)$ is irreducible if and only if $\mathrm{Tr}_{\mathbb{F}_{q^n}/\mathbb{F}_p}\left(\frac{-1}{\gamma+1}\right)\neq 0$. Given that,

$$H(x)=\sum_{i=0}^{n} h_i x^i = h_0+h_1x+h_2x^2+h_3x^3+\cdots\ldots+h_nx^n \tag{12.10}$$

Therefore,

$$\mathrm{Tr}_{q^n/p}(\gamma)=-\mathrm{Tr}_{q^n/p}\left(\frac{x+1}{x}\right)$$

Now,

$$H\left(\frac{x+1}{x}\right)=h_0+h_1\left(\frac{x+1}{x}\right)+h_2\left(\frac{x+1}{x}\right)^2+h_3\left(\frac{x+1}{x}\right)^3+\cdots\ldots+h_n\left(\frac{x+1}{x}\right)^n. \tag{12.11}$$

Since,

$$\mathrm{Tr}_{q^n/\mathrm{p}}\left(\frac{-1}{\gamma+1}\right)=\mathrm{Tr}_{q/\mathrm{p}}\left(\mathrm{Tr}_{q^n/\mathrm{q}}\left(\frac{-1}{\gamma+1}\right)\right)$$

and

$$\mathrm{Tr}_{q^n/\mathrm{q}}\left(\frac{-1}{\gamma+1}\right)=\frac{H'(1)}{H(1)}$$

$$\mathrm{Tr}_{q^n/\mathrm{p}}\left(\frac{-1}{\gamma+1}\right)=-\mathrm{Tr}_{q/\mathrm{p}}\left(\frac{H'(1)}{H(1)}\right)$$

Hence, $H(x)$ is an IRP over $\mathbb{F}_q$ if and only if

$$\mathrm{Tr}_{q/\mathrm{p}}\left(\frac{H'(1)}{H(1)}\right)\neq 0$$

Example 12.3.3 Let $g(x)=x^3+x+1$ be an IRP having root a such that $\mathbb{F}_{2^3}=\mathbb{F}_2(a)$ and

$$h(x)=ax^3+\left(a^2+1\right)x^2+(a+1) \qquad (12.12)$$

is an IRP over $\mathbb{F}_{2^3}$. Now,

$$h'(x)=3ax^2+2\left(a^2+1\right)x$$

$$=ax^2$$

So

$$h'(1)=a \text{ and } h(1)=a^2$$

Therefore,

$$\begin{aligned}\mathrm{Tr}_{2^3/2}\left(\frac{h'(1)}{h(1)}\right) &= \mathrm{Tr}_{2^3/2}\left(\frac{a}{a^2}\right)\\ &= \mathrm{Tr}_{2^3/2}\left(a^2+1\right)\\ &= \left(a^2+1\right)+\left(a^2+1\right)^2+\left(a^2+1\right)^4\\ &= 1\neq 0\end{aligned}$$

Since all the conditions of Theorem 3.2 are satisfied, the polynomial

$$t(x)=(x-1)^3h\left(\frac{x^2-x+1}{x-1}\right)=(x-1)^3\left(a\left(\frac{x^2-x+1}{x-1}\right)^3+\left(a^2+1\right)\left(\frac{x^2-x+1}{x-1}\right)^2+(a+1)\right)$$

$$\begin{aligned}&= ax^{12}+\left(a^2+1\right)x^{11}+ax^{10}+(a+1)x^9+\left(a^2+1\right)x^8\\ &\quad+\left(a^2+1\right)x^7+x^6+\left(a^2+1\right)x^4+\left(a^2+a\right)x^3+ax^2+a\end{aligned}$$

is an IRP of degree 12 over $\mathbb{F}_{2^3}$.

Example 12.3.4. Let $g(x)=x^3+2x+1$ be an IRP having root a such that $\mathbb{F}_{3^3}=\mathbb{F}_3(a)$ and

$$h(x)=(a+1)x^3+(2a+1)x^2+(a+2)x+(a+1) \qquad (12.13)$$

is an IRP over $\mathbb{F}_{3^3}$. Now it is required to prove that

$$\begin{aligned} &\mathrm{Tr}_{3^3/3}\left(\frac{h'(1)}{h(1)}\right)\neq 0 \\ &h'(x)=(a+2)x+a+2 \end{aligned} \tag{12.14}$$

Now,

$$\begin{aligned} h'(1)&=2a+1\\ h(1)&=(a+1)+(2a+1)+(a+2)(a+1)=2a+2\\ \mathrm{Tr}_{3^3/3}&\left(\frac{2a+1}{2a+2}\right)\neq 0 \end{aligned}$$

Since all the conditions of Theorem 3.2 are satisfied, the polynomial

$$\begin{aligned} t(x)&=h\left(\frac{x^3-x^2+1}{x^2-1}\right)\left(x^2-1\right)^3\\ &=(a+1)\left(\frac{x^3-x^2+1}{x^2-1}\right)^3+(2a+1)\left(\frac{x^3-x^2+1}{x^2-1}\right)^2+(a+2)\left(\frac{x^3-x^2+1}{x^2-1}\right)+(a+1)\\ &=(a+1)x^9+(2a+1)x^8+(2a+1)x^6+2a+1 \end{aligned}$$

is an IRP of degree 9 over $\mathbb{F}_{3^3}$.

Theorem 12.3.3. Let $h(x)=\sum_{j=0}^{n} h_j x^j$ over $\mathbb{F}_q$ be an n degree IRP, such that

$$\mathrm{Tr}_{q/p}\left(\frac{h^{*'}(0)}{h^{*}(0)}\right)\neq 0 \text{ and } \mathrm{Tr}_{q/p}\left(\frac{h'(0)}{h(0)}\right)\neq 0 \tag{12.15}$$

Then, the polynomial

$$t(x)=\left(x^p-x\right)^{pn} h\left(\frac{\left(x^p-x+1\right)^p-\left(x^p-x+1\right)\left(x^p-x\right)^{p-1}}{\left(x^p-x\right)^p}\right)$$

is irreducible with degree p^2n over $\mathbb{F}_q$.

Proof. Suppose the polynomial $h(x)$ over $\mathbb{F}_q$ has a root $\beta\in\mathbb{F}_q$ and $g(x)=x^p-x-\beta$ is a trinomial. Then,

$$\mathrm{Tr}_{q^n/p}(\beta)=\mathrm{Tr}_{q/p}\left(\mathrm{Tr}_{q^n/q}(\beta)\right)$$

So,

$$\mathrm{Tr}_{q^n/p}(\beta) = \mathrm{Tr}_{q/p}\left(\frac{-h_{n-1}}{h_n}\right) \tag{12.16}$$

We have

$$h(x) = h_0 + h_1 x + h_2 x^2 \cdots + h_{n-1} x^{n-1} + h_n x^n \tag{12.17}$$

$$h^*(x) = h_n + h_{n-1} x + \cdots + h_1 x^{n-1} + h_0 x^n \tag{12.18}$$

$$h^{*'}(x) = h_{n-1} + h_{n-2} x + \cdots + h_1 (n-1) x^{n-2} + h_0 n x^{n-1}. \tag{12.19}$$

After substituting $x = 0$ in Equations (12.18) and (12.19), we get

$$h^{*'}(0) = h_{n-1} \text{ and } h^*(0) = h_n \tag{12.20}$$

Using Equation (12.20) in Equation (12.16), we get

$$\mathrm{Tr}_{q^n/p}(\beta) = -\mathrm{Tr}_{q/p}\left(\frac{h^{*'}(0)}{h^*(0)}\right) \neq 0$$

or

$$\mathrm{Tr}_{q/p}\left(\frac{h^{*'}(0)}{h(0)}\right) \neq 0.$$

Now from Proposition 2.2

$$g(x) = x^p - x - \beta \tag{12.21}$$

is irreducible in $\mathbb{F}_{q^n}[x]$. By Lemma 3.1

$$H(x) = \left(x^p - x\right)^p g\left(\frac{x^p - x + 1}{x^p - x}\right)$$

is irreducible if and only if

$$\mathrm{Tr}_{q^n/p}\left(\frac{g'(1)}{g(1)}\right) \neq 0 \qquad (12.22)$$

Now, from Equation (12.21)

$$g'(x) = px^{p-1} - 1 = -1 \text{ and } g(1) = -\beta \tag{12.23}$$

By using Equation (12.23) in Equation (12.22), $H(x)$ is irreducible if and only if

$$\mathrm{Tr}_{q^n/p}\left(\frac{1}{\beta}\right) \neq 0.$$

We know that $h(x)$ has root β So, $h^*(x)$ has root as $\frac{1}{\beta}$. Therefore,

$$\begin{aligned}\mathrm{Tr}_{q^n/p}\left(\frac{1}{\beta}\right) &= \mathrm{Tr}_{q/p}\left(\mathrm{Tr}_{q^n/q}\left(\frac{1}{\beta}\right)\right)\\ &= \mathrm{Tr}_{q/p}\left(\frac{-h_1}{h_0}\right)\\ &= -\mathrm{Tr}_{q/p}\left(\frac{h'(0)}{h(0)}\right) \neq 0\end{aligned}$$

So

$$H(x) = \left(x^p - x\right)^p g\left(\frac{x^p - x + 1}{x^p - x}\right)$$

is irreducible over $\mathbb{F}_{q^n}$.

Therefore,

$$\begin{aligned}H(x) &= \left(x^p - x\right)^p \left(\left(\frac{x^p - x + 1}{x^p - x}\right)^p - \left(\frac{x^p - x + 1}{x^p - x}\right) - \beta\right)\\ &= \left(x^p - x\right)^p \left(\frac{\left(x^p - x + 1\right)^p - \left(x^p - x\right)^{p-1}\left(x^p - x + 1\right) - \beta\left(x^p - x\right)^p}{\left(x^p - x\right)^p}\right)\\ &= \left(x^p - x + 1\right)^p - \left(x^p - x\right)^{p-1}\left(x^p - x + 1\right) - \beta\left(x^p - x\right)^p\end{aligned}$$

is an IRP over $\mathbb{F}_{q^n}$. Hence,

$$t(x) = \left(x^p - x\right)^{pn} h\left(\frac{\left(x^p - x + 1\right)^p - \left(x^p - x\right)^{p-1}\left(x^p - x + 1\right)}{\left(x^p - x\right)^p}\right)$$

is irreducible over $\mathbb{F}_q$ with degree $p^2 n$.

Example 12.3.5 Let $g(x) = x^3 + 2x + 1$ be an IRP having root a such that $\mathbb{F}_{3^3} = \mathbb{F}_3(a)$ and

$$h(x) = (a+1)x^3 + (2a+1)x^2 + (a+2)x + (a+1) \tag{12.24}$$

is an IRP over $\mathbb{F}_{3^3}$. Now it is required to prove that

$$\mathrm{Tr}_{3^3/3}\left(\frac{h'(0)}{h(0)}\right) \neq 0 \tag{12.25}$$

and

$$\mathrm{Tr}_{3^3/3}\left(\frac{h^{*'}(0)}{h^{*}(0)}\right) \neq 0 \tag{12.26}$$

From Equation (12.24)

$$h'(x) = 3(a+1)x^2 + 2(2a+1)x + (a+2) \tag{12.27}$$

Putting $x = 0$ in Equations (12.27) and (12.24),

$$h(0) = a+1 \text{ and } h'(0) = a+2. \tag{12.28}$$

Using Equation (12.28) in Equation (12.25), we get

$$\begin{aligned}\mathrm{Tr}_{3^3/3}\left(\frac{h'(0)}{h(0)}\right) &= \mathrm{Tr}_{3^3/3}\left(\frac{a+2}{a+1}\right)\\ &= \mathrm{Tr}_{3^3/3}\left(a^{20}\right) = 1 \neq 0\end{aligned}$$

The reciprocal polynomial of Equation (12.24) is

$$h^{*}(x) = (a+1)x^3 + (a+2)x^2 + (2a+1)x + (a+1) \tag{12.29}$$

and

$$h^{*'}(x) = 3(a+1)x^2 + 2(a+2)x + (2a+1) \tag{12.30}$$

Substituting $x = 0$ in Equations (12.29) and (12.30), we get

$$h^{*}(0) = a+1 \text{ and } h^{*'}(0) = 2a+1. \tag{12.31}$$

Therefore, putting the values of Equation (12.31) in Equation (12.26), we have

$$\begin{aligned}\mathrm{Tr}_{3^3/3}\left(\frac{h^{*'}(0)}{h^{*}(0)}\right) &= \mathrm{Tr}_{3^3/3}\left(\frac{2a+1}{a+1}\right)\\ &= \mathrm{Tr}_{3^3/3}\left(a^{7}\right) = 2 \neq 0\end{aligned}$$

Since all the conditions of Theorem 3.3 are satisfied, the polynomial over $\mathbb{F}_{3^3}$ is a 27-degree IRP.

$$t(x)=\left(x^3-x\right)^9 h\left(\frac{\left(x^3-x+1\right)-\left(x^3-x\right)^2\left(x^3-x+1\right)}{\left(x^3-x\right)^3}\right)$$

$$=\left(x^3-x+2\right)^9 h\left(\frac{2x^6-4x^4+3x^3+2x^2-3x+1}{\left(x^3-x\right)^3}\right)$$

$$=\left(x^3-x+2\right)^9 h\left(\frac{2x^6-x^4+2x^2+1}{\left(x^3-x\right)^3}\right)$$

$$=\left(x^3-x+2\right)^9\left((a+1)\left(\frac{2x^6-x^4+2x^2+1}{\left(x^3-x\right)^3}\right)^3+(2a+1)\left(\frac{2x^6-x^4+2x^2+1}{\left(x^3-x\right)^3}\right)^2\right.$$

$$\left.+(a+2)\left(\frac{2x^6-x^4+2x^2+1}{\left(x^3-x\right)^3}\right)+(a+1)\right)$$

$$=(a+1)x^{27}+(2a+1)x^{24}+(2a+1)x^{22}+(2a+1)x^{21}+(2a+1)x^{20}+(a+2)x^{19}$$
$$+(2a+2)x^{18}+(a+2)x^{15}+(2a+2)x^{12}+(2a+1)x^{11}+(2a+1)x^{10}+ax^9$$
$$+(2a+1)x^{16}+(2a+1)x^8+(2a+1)x^7+x^6+(a+2)x^5+(a+2)x^3+a+1 \quad .$$

12.4 CONSTRUCTION OF NPS OVER FINITE FIELDS

Theorem 12.4.1. Let $h(x)=\sum_{j=0}^{n} h_j x^j$ over $\mathbb{F}_q$ be an n degree IRP and $h^*(x)=K(x)$ such that $K(x)$ and $K^*(x+1)$ are NPs over $\mathbb{F}_q$. Also consider the polynomial

$$H(x)=\left(x^p-x\right)^n h\left(\frac{x^p-x+1}{x^p-x}\right) \tag{12.32}$$

such that

$$\mathrm{Tr}_{q/p}\left(\frac{h'(1)}{h(1)}\right)\neq 0$$

then $H^*(x)$ over $\mathbb{F}_q$ is an NP with degree pn.

Proof. Suppose $\beta \in \mathbb{F}_{q^n}$ is a root of $h(x)$ and $\mathrm{Tr}_{q/p}\left(\frac{h'(1)}{h(1)}\right) \neq 0$. So, the polynomial $H(x)$ over $\mathbb{F}_q$ is irreducible by Lemma 3.1. Thus, the polynomial $H^*(x)$ over $\mathbb{F}_q$ is also irreducible.

Now, write $n = n_1 p^e$, where $n_1 \in \mathbb{N}$ with $\gcd(n_1, p) = 1$ such that $e \geq 0$. Let $p^e = t$ and $x^n - 1$ can be factorized in $\mathbb{F}_q$ [x] as:

$$x^n - 1 = \left(x^{n_1} - 1\right)^t = \left(l_1(x) l_2(x) \cdots l_k(x)\right)^t \tag{12.33}$$

where $l_i(x) \neq l_k(x)$ and are all irreducible.

$$\text{Define, } p_i(x) = \frac{x^n - 1}{l_i(x)} = \sum_{u=0}^{v_i} c_{iu} x^u, i = 0,1,2,3,\ldots,k \tag{12.34}$$

where $m_i = n - \deg\left(l_i(x)\right)$ and $c_{iu} \epsilon \mathbb{F}_q$.

Let $H(x)$ have root α_1, and then $\beta_1 = \frac{1}{\alpha_1}$ will be the root of $H^*(x)$. Further, we prove that the polynomial $H^*(x)$ is normal. Now, using Equation (12.33) we have

$$x^{pn} - 1 = \left(l_1(x) l_2(x) \cdots l_k(x)\right)^{pt}$$

Consider

$$\begin{aligned} T_i(x) &= \frac{x^{pn} - 1}{l_i(x)} \\ &= \frac{x^n - 1}{l_i(x)} \left(\sum_{t=0}^{p-1} x^{tn} \right), i = 0,1,2,\ldots k \end{aligned}$$

Now, from Equation (12.34)

$$T_i(x) = \sum_{u=0}^{m_i} c_{iu} \left(\sum_{t=0}^{p-1} x^{tn+u} \right)$$

Thus, from Proposition 2.4

$$L_{T_i}(\beta_1) = \sum_{u=0}^{m_i} c_{iu} \left(\sum_{t=0}^{p-1} (\beta_1)^{p^{tsn}} \right)^{p^{su}}$$

$$= \sum_{u=0}^{m_i} c_{iu} \left(\sum_{t=0}^{p-1} \left(\frac{1}{\alpha_1} \right)^{p^{tsn}} \right)^{p^{su}} \tag{12.35}$$

Now from Equation (12.32), if $H(x)$ has root α_1, then $\frac{\alpha_1^p - \alpha_1 + 1}{\alpha_1^p - \alpha_1}$ is also a root of $h(x)$. So

$$\beta = \frac{\alpha_1^p - \alpha_1 + 1}{\alpha_1^p - \alpha_1} \tag{12.36}$$
$$\beta = \left(\alpha_1^p - \alpha_1 + 1\right)\left(\alpha_1^p - \alpha_1\right)^{-1}.$$

Since β is root $h(x)$, we have

$$\begin{aligned}\beta = \beta^{p^{sn}} &= \left(\alpha_1^p - \alpha_1 + 1\right)^{p^{sn}} \left(\alpha_1^p - \alpha_1\right)^{-p^{sn}} \\ &= \left(\alpha_1^p - \alpha_1 + 1\right)\left(\alpha_1^p - \alpha_1\right)^{-1}\end{aligned}$$

Therefore, from Equation (12.36)

$$\frac{\left(\alpha_1^p - \alpha_1 + 1\right)^{p^{sn}}}{\left(\alpha_1^p - \alpha_1\right)^{p^{sn}}} = \frac{\left(\alpha_1^p - \alpha_1 + 1\right)}{\left(\alpha_1^p - \alpha_1\right)}$$

Since $\left(\alpha_1^p - \alpha_1\right) \neq 0$,

$$\frac{\left(\alpha_1^{p^{sn+1}} - \alpha_1^{p^{sn}} + 1\right)}{\alpha_1^{p^{n+1}} - \alpha_1^{p^{sn}}} = \frac{\left(\alpha_1^p - \alpha_1 + 1\right)}{\left(\alpha_1^p - \alpha_1\right)}.$$

Thus, we have

$$\alpha_1^{p^{sn+1}} - \alpha_1^{p^{sn}} + 1 = \alpha_1^p - \alpha_1 + 1$$

or

$$\alpha_1^{p^{sn+1}} - \alpha_1^p = \alpha_1^{p^{sn}} - \alpha_1$$

or

$$\left(\alpha_1^{p^{sn}} - \alpha_1\right)^p = \alpha_1^{p^{sn}} - \alpha_1. \tag{12.37}$$

Now, it follows from Equation (12.37) if $\alpha_1^{p^{sn}} - \alpha_1 = \delta$, then $\alpha_1^{p^{sn}} = \alpha_1 + \delta$, and we can verify that

$$\alpha_1^{p^{tsn}} = \alpha_1 + t\delta, 0 \le t \le p-1.$$

Or

$$\left(\frac{1}{\alpha_1}\right)^{p^{tsn}} = \frac{1}{\alpha_1 + t\delta}. \tag{12.38}$$

Using Equation (12.37) in Equation (12.35)

$$L_{T_i}(\beta_1) = \sum_{u=0}^{m_i} c_{iu} \left(\sum_{t=0}^{p-1} \frac{1}{\alpha_1 + t\delta}\right)^{p^{su}}.$$

From Proposition 2.3,

$$L_{T_i}(\beta_1) = \sum_{u=0}^{m_i} c_{iu} \left(\frac{-1}{\alpha_1^p - \alpha_1}\right)^{p^{su}}$$

From Equation (12.36), we get

$$L_{T_i}(\beta_1) = -\sum_{u=0}^{m_i} c_{iu} \left(\frac{\beta}{1-\beta}\right),$$

it becomes

$$L_{T_i}(\beta_1) = -L_{p_i}\left(\frac{\beta}{1-\beta}\right) \tag{12.39}$$

Since $h^*(x) = K(x)$ has root $\frac{1}{\beta}$, $\left(\frac{1}{\beta} - 1\right)$ is a root of $K(x+1)$. Thus $K^*(x+1)$ has root $\frac{\beta}{1-\beta}$. Since $K^*(x+1)$ is an NP, using Proposition 2.2

$$L_{pi}\left(\frac{\beta}{1-\beta}\right) \neq 0, i = 0,1,2,\ldots,k.$$

It follows from Equation (12.39)

$$L_{pi}(\beta_1) \neq 0$$

for each $i = 0,1,2,3,\ldots k$. Thus, β_1 over $\mathbb{F}_q$ is a normal element from Proposition 2.4. Hence, the polynomial $H^*(x)$ is normal over $\mathbb{F}_q$.

Example 12.4.1. Assume $g(x) = x^3 + 2x + 1$ is an IRP having root a satisfying $\mathbb{F}_{3^3} = \mathbb{F}_3(a)$ and

$$h(x) = (a+1)x^3 + (2a+1)x^2 + (a+1)x + (a+1) \tag{12.40}$$

is irreducible over $\mathbb{F}_{3^3}$. Then

$$h^*(x) = (a+1)x^3 + (a+1)x^2 + (2a+1)x + (a+1) = K(x)$$

and

$$K(x+1) = (a+1)x^3 + (a+1)x^2 + ax + 2a + 1$$
$$K^*(x+1) = (2a+1)x^3 + ax^2 + (a+1)x + a + 1$$

Now, we will show that

$$\mathrm{Tr}_{3^3/3}\left(\frac{h'(1)}{h(1)}\right) \neq 0$$

That is,

$$\mathrm{Tr}_{3^3/3}\left(\frac{h'(1)}{h(1)}\right) \neq 0 \tag{12.41}$$

Now, using the derivative of Equation (12.40), we get

$$h'(x) = 3(a+1)x^2 + 2(2a+1)x + (a+1) = (a+2)x + (a+2), \tag{12.42}$$

put $x = 0$ in Equation (12.42),

$$h'(1) = a \text{ and } h(1) = 2a + 1 \tag{12.43}$$

Using the values of Equation (12.43) in Equation (12.41), we have

$$\begin{aligned}\mathrm{Tr}_{3^3/3}\left(\frac{h'(1)}{h(1)}\right) &= \mathrm{Tr}_{3^3/3}\left(\frac{a}{2a+1}\right)\\ &= \mathrm{Tr}_{3^3/3}\left(\frac{a}{a^{16}}\right)\\ &= \mathrm{Tr}_{3^3/3}\left(a^{11}\right)\\ &= 1 \neq 0.\end{aligned}$$

Here, the polynomial

$$\begin{aligned} H(x) &= \left(x^3 - x\right)^3 h\left(\frac{x^3 - x + 1}{x^3 - x}\right) \\ &= \left(x^3 - x + 2\right)^3 \left(\begin{array}{l} (a+1)\left(\frac{x^3 - x + 1}{x^3 - x}\right)^3 + (2a+1)\left(\frac{x^3 - x + 1}{x^3 - x}\right)^2 \\ +(a+1)\left(\frac{x^3 - x + 1}{x^3 - x}\right) + (a+1) \end{array}\right) \\ &= (2a+1)x^9 + 2ax^6 + 2ax^4 + 2ax^2 + (a+2)x + (a+1). \end{aligned}$$

Since the conditions of the previous theorem are satisfied, the polynomial

$$H^*(x) = (a+1)x^9 + (a+2)x^8 + 2ax^7 + 2ax^5 + 2ax^3 + (2a+1)$$

is a normal over $\mathbb{F}_{3^3}$.

12.5 CONCLUSION

In this chapter, an alternative method for constructing IRPs and NPs is introduced by utilizing the transformation $x \mapsto \frac{x^p - x + 1}{x^p - x}$, examples of IRPs are constructed using Mathematica 9.3 and SageMath 9.3.

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Chapter 13

Enumeration of Latin Squares Constructed Using Bivariate Permutation Polynomials over Finite Fields

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13.1 INTRODUCTION

In many domains nowadays, including the military, medical, education, and personal affairs, a large number of digital photos are produced. Image encryption has become a popular topic in academic study and its practical application since digital photos often include secret or private information. Among the suggested algorithms for image encryption, a significant percentage are based on chaos. The application of chaos to cryptography has been the subject of increasing research over the past few decades. Encryption algorithms and chaotic systems share a natural connection in their emphasis on confusion and diffusion. Chaotic systems exhibit cryptographic properties like ergodicity and sensitivity to initial conditions. Despite having many strong cryptographic features, many of the chaotic image encryption algorithms nowadays in use have flaws in the following areas, regardless of the chaotic systems being used. To overcome the shortcomings of chaotic image ciphers, such as round-off errors and randomness degradations, Latin squares (LSs), self-orthogonal Latin squares (SOLSs), and mutually orthogonal Latin squares (MOLSs) were introduced. An LS is a particular kind of square matrix with uniformity. In other words, L is called an LS of order n if each column and each row contains every element of the set of entries of the square exactly once.

Shannon (1949) was the first to draw the link between encryption and LSs in his seminal study. Some excellent features of the LS make it ideal for image encryption. Euler identifies the study of LSs as having its origins in the root of the 36-officer problems, which deal with the placement of 36 officers in 6*6 squares. Euler (1782) initially conjectured that, for n congruent to 2 modulo 4, the maximum number of LSs in a set of MOLSs of side n is 1. Later, MacNeish (1922) made the error of thinking he had validated Euler's conjecture, leading him to propose a stronger one. It was shown by Bose (1938) that, for any natural number or prime number, a certain number of points and lines can be constructed using the properties of a Galois field. Furthermore, it was established that the existence of such a projective geometry is the same as the existence of n-sided orthogonal LSs. Later, Parker (1959, 1962) came up with a new construction method based on balanced incomplete block designs. Further, Bose and Shrikhande (1960) generalized Parker's work by replacing balanced incomplete block designs with a partially balanced design.

A recent study shows that mutually unbiased bases (MUBs) play a significant role in various fields, particularly in quantum cryptography, where they are instrumental in designing quantum key distribution protocols. MUBs can be constructed using MOLSs. Specifically, for MOLSs of order (n), a resolvable block design (RBD) with (n^2) entities can be derived in a dimension $(d = n^2)$. Each MOLSs of order (n) contributes MOLSs $(n)+2$ parallel classes to the RBD of order n^2. These parallel classes are then transformed into orthonormal bases using the Hadamard matrix of order n. If a real Hadamard matrix of order n exists, it enables the construction of real MUBs.

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Groups and quasigroups are important in information security, which is the study and practice of preventing unauthorized access, use, alteration, or destruction of data. Computational security in cryptographic systems is greatly influenced by the algebraic structures and operations that underlie algorithms and how they are implemented. Finite groups and Galois fields serve as the foundational elements of algebraic structures employed in the design of cryptographic systems. Therefore, a comprehensive examination of the underlying finite groups and Galois fields using both scientific and mathematical methodologies is necessary to determine the level of security and robustness that such a system offers. Since any finite group structure can be represented by a Cayley table, and a Galois field structure can be represented by two Cayley tables, evaluating a given finite group or Galois field representative as an LS may yield many important properties. A list of n unique symbols, one for each row and column, is a transversal of an LS.

Schmidt (2016) examined the use of LSs in cryptography and then went on to relate Cayley tables and LSs to permutations and permutation groups. Author generated LSs and counted their transversals. Moreover, he created an algorithm to construct prime order $-p$ cyclic LSs as "building blocks" to construct super-symmetric LSs of prime power order $-p^d$ with $d>0$.

Authors of the studies (Bhatta, 2021; Bhatta & Shankar, 2009; Rivest, 2001; Shankar & Bhatta, 2009) and Bhatta et al. (2022) discussed LSs formed using permutation polynomials (PPs) in various aspects. In (Mehendale, 2006; RAS, 2020; Samardjiska & Gligoroski, 2017 and Donovan et al. (2020), MOLSs are extensively studied. Authors used symmetry-breaking strategies based on certain characteristics of diagonal LSs. Bivariate PPs and the existence of LSs are equivalent in any finite field. On this note, Chen et al. (2023) constructed four classes of bivariate PPs considered over the finite field of even characteristics. Vishwakarma and Singh (2024) discussed methods to use PPs considered over finite fields to create MOLSs. Colbourn and Dinitz (2001) provided more information regarding the construction of LS. For more information on the LSs, refer to (Laywine & Mullen, 1998; Olsson, 2017; Wilson, 1974; Zolfaghari & Bibak, 2022).

Any function defined over a finite field can be represented by a polynomial whose coefficients reside in that same field. Lagrange's interpolation formula provides us with a way to construct such a polynomial. If L is the LS of order $q=p^n$, where p is prime, then each entry will be denoted by s_{ij} for $\left(x_i,y_j\right)\in F_q^2$. Let $f:F_q^2\to F_q$ be a bijective function such that $f\left(x_i,y_j\right)=s_{ij}$ for $1\le i,j\le q-1$ and $s_{ij}\in F_q$ is a symbol corresponding to $\left(x_i,y_j\right)$. Then, an LS can be represented in terms of a bivariate polynomial $H\left(x,y\right)$ using the function f over the finite field F_q^2 as follows:

$$H(x,y)=\sum_{\left(x_i,y_j\right)\in F_q^2} f\left(x_i,y_j\right)\left(1-\left(x-x_i\right)^{q-1}\right)\left(1-\left(y-y_j\right)^{q-1}\right).$$

The aim of this chapter is to construct LSs of order q using bivariate PPs over the finite field F_q. More specifically, it is observed that all 576 LSs of order 4 found in the finite field F_{2^2} can be expressed as

$$a+bx+cy+dx^2+ey^2+fxy+gx^2y+hxy^2+ix^2y^2 \quad (13.1)$$

with certain conditions on the coefficients. And the same polynomial works for all the finite fields of order 2^n. Additionally, all LSs of these types are enumerated over the finite field F_{2^n}. Furthermore, symmetric and idempotent LSs of order 2^n in the finite field F_{2^n} are characterized and enumerated. Moreover, it is observed that each specified polynomial gives a complete set of $q-1$ MOLSs of order $q=2^n$. In addition, the existence of mutually orthogonal and SOLSs for the specified polynomial over F_{2^2} is identified.

Organization of the Chapter

The rest of the chapter is organized as follows. Section 13.2 enlightens preliminaries, and Section 13.3 proves the existence of bivariate PPs for LSs in the arbitrary finite field F_q and their orthogonality. Section 13.4 proposes a generalized bivariate PP that gives all the 576 LSs of order 4 in the finite field F_{2^2}. Notice that the same polynomial works for all the finite fields of order 2^n. Hence, for the same polynomial, characterization and enumeration of the existence of symmetric and idempotent LSs of order 2^n in the finite field F_{2^n} are listed in Section 13.5. At last, mutually orthogonal and SOLSs of specified polynomials are characterized in Section 13.6. Finally, Section 13.7 concludes the chapter.

13.2 PRELIMINARIES

In this section, all the necessary definitions and supporting results to generate LSs using bivariate PPs over the finite field F_q are listed. The definitions of LSs, orthogonal LSs, and MOLSs have been taken from Van Lint et al. (2001).

Theorem 13.2.1. (Van Lint et al., 2001) Let $a_0, a_1, a_2, \ldots, a_{q-1}$ be the elements of $\mathbb{F}_q$, then the arrays

$$L_k = \begin{bmatrix} a_0 & a_1 & \cdots & a_{q-1} \\ a_k a_1 & a_k a_2 + a_1 & \cdots & a_k a_1 + a_{q-1} \\ \cdot & \cdot & \cdots & \cdot \\ a_k a_{q-1} & a_k a_{q-1} + a_1 & \cdots & a_k a_{q-1} + a_{q-1} \end{bmatrix}$$

$k = 1, \ldots, q-1$ form a set of $q-1$ MOLSs of order q, where $a_j^k = a_k a_{i-1} + a_{j-1}$ is (i,j) entry of L_k.

LSs are considered a representation of bivariate PPs with the help of the following definition.

Definition 13.2.2. (Rivest, 2001) Let $H(x,y) = \sum_{i,j} a_{ij} x^i y^j$ represent a bivariate polynomial. Then there exists a function $H : S^2 \to S$ on a finite set S of size $n > 1$; it is said to be an LS if for any $a, b \in S$, both functions $H(a,y), H(x,b)$ are permutation functions of S.

Definition 13.2.3. (Lidl & Niederreiter, 1997) The character function $\chi_1(c) = e^{\frac{2\pi i Tr(c)}{p}}$ defined on the additive group of F_q is known as the canonical additive character of F_q.

Definition 13.2.4. (Lidl & Niederreiter, 1997) The function χ_b with $\chi_b(c) = \chi_1(bc)$ for all $b, c \in F_{q^n}$ is known as the additive character of F_{q^n}.

By setting $b = 0$ in the previous definition, one can get a trivial additive character χ_0, such that $\chi_0(c) = 1$ for all $c \in F_{q^n}$.

The number of solutions of a polynomial function and properties of additive characters are interrelated to observe the permutation behavior of polynomials. The following results are stated in Lidl and Niederreiter (1997), which are readily used in this chapter.

Theorem 13.2.5. (Lidl & Niederreiter, 1997) The polynomial $H(x) \in F_q[x_1,\ldots,x_n]$ is a PP over F_q if and only if

$$\sum_{(c_1,\ldots,c_n)\in F_q^n} \chi\big(H(c_1,\ldots,c_n)\big) = 0,$$

for all non-trivial additive characters χ of F_q.

Theorem 13.2.6. (Lidl & Niederreiter, 1997) If χ is a non-trivial additive character of the finite field F_{q^n} then $\sum_{H\in F_{q^n}} \chi(H) = 0$.

Remark 13.2.7. (Lidl & Niederreiter, 1997) For any $d \in F_{q^n}$

$$N = \frac{1}{q}\sum_{c\in F_{q^n}}\sum_{\chi}\chi\big(H(c)\big)\overline{\chi(d)}, \tag{13.2}$$

is the number of solutions of $H(x) = d$ in F_{q^n}.

Theorem 13.2.8. (Lidl & Niederreiter, 1997) Let $f \in \mathbb{F}_q[x], g \in \mathbb{F}_q[y]$. Suppose a total number of roots of $f(x) = 0$ in $\mathbb{F}_q$ is not divisible by q. Let $h \in \mathbb{F}_q[x,y]$ be such that $h(c,y)$ is PP considered over $\mathbb{F}_q$ in $y, \forall c \in \mathbb{F}_q$. Then, $H(x,y) = h(x,y)f(x) + g(y)$ is not a bivariate PP considered over $\mathbb{F}_q$.

In the following section, some special kinds of bivariate PPs that form LS over any finite field $\mathbb{F}_q$ are characterized, more specifically, their orthogonality.

13.3 EXISTENCE OF LSS IN $\mathbb{F}_q$ AND THEIR ORTHOGONALITY

Definition 13.3.1. *A system of two bivariate PPs H_1 and H_2 in $\mathbb{F}_q[x]$ are said to be orthogonal over $\mathbb{F}_q$ if the system of equations $H_1(x,y) = a_1, H_2(x,y) = a_2$ has only one root in $\mathbb{F}_q^2$ for each $(a_1,a_2) \in \mathbb{F}_q^2$. In particular, the corresponding LSs of these bivariate PPs H_1 and H_2 in $\mathbb{F}_q[x]$ are orthogonal to each other.*

Definition 13.3.2. *A set of bivariate PPs corresponding to each LS is called a set of mutually orthogonal PPs if the corresponding LS of each PP is orthogonal in $\mathbb{F}_q[x]$.*

Lemma 13.3.3. *Let $H(x,y) = ax^2 + by^2 + cxy$ be a bivariate polynomial considered over $\mathbb{F}_q$ where $a \cdot b \cdot c \neq 0$, then it is not an LS over $\mathbb{F}_q$.*

Proof. It must be shown that $H(x,y) = ax^2 + by^2 + cxy$ is not a PP considered over $\mathbb{F}_q$. Here, $h(x,y) = ax + cy, f(x) = x$ and $g(y) = by^2$. The total number of roots of $f(x) = x$ in $\mathbb{F}_q$ is 1, which is not divisible by q. And $h(d,y) = ad + cy$, which is a linear polynomial considered over $\mathbb{F}_q$; hence, it is a PP considered over $\mathbb{F}_q$. Thus, from Theorem (13.2.8), $H(x,y)$ is not a bivariate PP considered over $\mathbb{F}_q$.

Theorem 13.3.4. *Let* $H(x,y) = ax^{k_1} + by^{k_2}$ *be a bivariate polynomial considered over* $\mathbb{F}_q$ *with* $a \neq 0$ *and* $b \neq 0$. *Then, the corresponding square is LS if and only if* $\gcd(k_i, q-1) = 1$ *for* $i = 1,2$.

Proof. Consider $H(x,y) = ax^{k_1} + by^{k_2}$; it may be expressed as $H(x,y) = g(x) + h(y)$. It must be shown that $H(x,y)$ is a PP if any one of the polynomials $g(x)$ or $h(y)$ is a PP considered over $\mathbb{F}_q$. Here, both $g(x)$ and $h(y)$ are monomials in x and y, respectively. It is known that monomials are PPs if and only if $\gcd(k_i, q-1) = 1$ for $i = 1,2$.

Theorem 13.3.5. *Let* L *be an LS in* $\mathbb{F}_q$ *and* H *be the corresponding bivariate PP considered over* $\mathbb{F}_q$. *Then, the square corresponding to the polynomial* $bH + c$, *where* $b \in \mathbb{F}_q^*$ *also forms an LS in* $\mathbb{F}_q$ *for each* $c \in \mathbb{F}_q$.

Proof. It must be shown that $bH + c$ is a bivariate PP considered over $\mathbb{F}_q$.

$$\sum_{(c_1,c_2)\in\mathbb{F}_q^2} \chi\big(bH(c_1,c_2) + c\big) = \sum_{(c_1,c_2)\in\mathbb{F}_q^2} \chi_b\big(H(c_1,c_2)\big)\chi(c),$$

because $H(x,y)$ is a PP considered over $\mathbb{F}_q$, from Theorem (13.2.5),

$$\sum_{(c_1,c_2)\in\mathbb{F}_q^2} \chi\big(H(c_1,c_2)\big) = 0,$$

for all non-trivial additive characters. Hence,

$$\sum_{(c_1,c_2)\in\mathbb{F}_q^2} \chi\big(bH(c_1,c_2) + c\big) = 0.$$

Theorem 13.3.6. *Let* $\{L_1,\ldots,L_k\}$ *be the set of LS over* $\mathbb{F}_q$ *and* $\{H_1,\ldots,H_k\}$ *be the set of corresponding bivariate PPs of each* L_i *over* $\mathbb{F}_q$. *Then, any two LS s,* L_i *and* L_j, *are orthogonal over* $\mathbb{F}_q$ *for* $1 \le i,j \le k$ *if and only if,* $\forall (b_1,b_2) \in \mathbb{F}_q^2$ *with* $(b_1,b_2) \neq (0,0)$ *the polynomial* $b_1H_i + b_2H_j$ *is a PP considered over* $\mathbb{F}_q$.

Proof. If any two LS L_i and L_j are orthogonal over $\mathbb{F}_q$ for $1 \le i,j \le k$, then

$$\sum_{(c_1,c_2)\in\mathbb{F}_q^2} \chi_{b_1}\big(H_i(c_1,c_2)\big)\chi_{b_2}\big(H_j(c_1,c_2)\big) = 0.$$

It is known that χ is additive character in $\mathbb{F}_q$, hence

$$\begin{aligned}\chi_{b_1}\big(H_i(c_1,c_2)\big)\chi_{b_2}\big(H_j(c_1,c_2)\big) &= \chi_1\big(b_1H_i(c_1,c_2)\big)\chi_1\big(b_2H_j(c_1,c_2)\big)\\ &= \chi_1\big(b_1H_i(c_1,c_2) + b_2H_j(c_1,c_2)\big).\end{aligned}$$

So,

$$\sum_{(c_1,c_2)\in\mathbb{F}_q^2} \chi_1\big(b_1H_i(c_1,c_2) + b_2H_j(c_1,c_2)\big) = 0.$$

Hence, from Theorem (13.2.5) $b_1H_i(x,y)+b_2H_j(x,y)$ is a PP considered over $\mathbb{F}_q$. Conversely, assume $b_1H_i(x,y)+b_2H_j(x,y)$ is a PP considered over $\mathbb{F}_q$, then it has q roots in $\mathbb{F}_q^2$. Now, it is needed to prove that any two LSs L_i and L_j are orthogonal over $\mathbb{F}_q$ for $1\le i,j\le k$. Equivalently, it is needed to prove that $N(a_1,a_2)=1$ for any $a_1,a_2\in\mathbb{F}_q^2$.

$$\begin{aligned}N(a_1,a_2)&=\frac{1}{q^2}\sum_{b_1\in\mathbb{F}_q}\chi_1\left(b_1H_i(c_1,c_2)\right)\overline{\chi_{b_1}(a_1)}\sum_{b_2\in\mathbb{F}_q}\chi_1\left(b_2H_j(c_1,c_2)\right)\overline{\chi_{b_2}(a_2)}\\&=\frac{1}{q^2}\sum_{(b_1,b_2)\in\mathbb{F}_q^2}\overline{\chi_{b_1}(a_1)\chi_{b_2}(a_2)}\sum_{(c_1,c_2)\in\mathbb{F}_q^2}\chi_{b_1}\left(H_i(c_1,c_2)\right)\chi_{b_2}\left(H_j(c_1,c_2)\right)\\&=\frac{1}{q^2}\cdot q^2\\&=1.\end{aligned}\tag{13.3}$$

Remark 13.3.7. *(Samardjiska & Gligoroski 2017) considered two quasigroups Q_1 and Q_2, which are constructed using complete mappings H_1 and H_2, and they proved that Q_1 and Q_2 are mutually orthogonal if and only if H_1+H_2 is a PP considered over $\mathbb{F}_{2^n}$.*

13.4 LSS OF ORDER 2^n

Let $\mathbb{F}_p$ be a prime field with p elements and $H(x)$ be the nth degree monic irreducible polynomial considered over $\mathbb{F}_p[x]$, then $\mathbb{F}_p[x]/H(x)$ is an extension field of $\mathbb{F}_p$. If $\mathbb{F}_2$ is the prime field of characteristic 2, then the extension field $\mathbb{F}_{2^2}$ will be constructed by considering $H(x)$ as a two-degree irreducible monic polynomial. So elements of $\mathbb{F}_{2^2}$ are $\{0,1,\alpha,\alpha+1\}$ if α is the root of $H(x)$.

Here, if bivariate PP is $H(x,y)=ax+by$, where $a,b\in\mathbb{F}_{2^2}$, then a total of 16 possibilities will arise, among which only nine form LS. For example, if α is a root of x^2+x+1, then $\mathbb{F}_{2^2}=\{0,1,\alpha,\alpha+1\}$. By taking $H(x,y)=(\alpha+1)x+y$, the corresponding LS is listed in Table 13.1, so that one can make a list of nine different combinations of $a,b\in\mathbb{F}_{2^2}$ to form an LS by $H(x,y)$, which can be seen in Table 13.2.

For a variety of reasons, MOLSs have attracted research interest. It is evident that up to $n-1$ MOLSs exist for order n. Let $N(n)$ stand for the maximum possible number of MOLSs of order n. It is proven that for each $n\ge 2, N(n)\le n-1$. Consequently, the set of $n-1$ MOLSs of order n is called complete. It is familiar with the fact that a set of MOLSs of order prime power is always complete. Moreover, for a prime power q, the set of

Table 13.1 LS of $H(x,y)=(\alpha+1)x+y$

$y\setminus x$	0	1	α	$\alpha+1$
0	0	$\alpha+1$	1	α
1	1	α	0	$\alpha+1$
α	α	1	$\alpha+1$	0
$\alpha+1$	$\alpha+1$	0	α	1

Table 13.2 Possible *a* and *b* in *H*

	a	*b*
L_1	1	1
L_2	α	α
L_3	$\alpha+1$	$\alpha+1$
L_4	1	α
L_5	α	1
L_6	α	$\alpha+1$
L_7	$\alpha+1$	α
L_8	$\alpha+1$	1
L_9	1	$\alpha+1$

polynomials of the type $x+ay$ with $a\in\mathbb{F}_q^*$ gives a complete set of $q-1$ MOLSs of order q (see C. F. Laywine and Mullen 1998). In this chapter, it is observed that each specified polynomial gives a complete set of $q-1$ MOLSs of order q.

Example 13.4.1. *Consider the MOLS of order 4 from Table 13.3.*

Here, the collections $S_1=\{L_1,L_4,L_5\}$ and $S_2=\{L_1,L_4,L_8\}$ are complete.

In the literature, one can see that there are 576 LSs of order 4. And it is noticed that the bivariate polynomials corresponding to all the 576 LSs are in the form of

$$a+bx+cy+dx^2+ey^2+fxy+gx^2y+hxy^2+ix^2y^2,$$

with certain conditions on the coefficients. Moreover, the same polynomials generate all the required LSs of order $q=2^n$ in any finite field $\mathbb{F}_q$ with certain conditions on the coefficients. Hence, all the LSs of these types of orders $q=2^n$ are characterized and enumerated.

Proposition 13.4.2. *Let $H(x,y)=ax+by+c$ be a bivariate polynomial considered over $\mathbb{F}_q$; then it is an LS if and only if $a\cdot b\neq 0$. There exist q^3-2q^2+q LSs of order q over $\mathbb{F}_q$.*

Proof. It must be shown that $H(x,y)=ax+by+c$ is a bivariate PP considered over $\mathbb{F}_q$; that is, $H(x,y)=k$ has q roots in $\mathbb{F}_q$.

$$\begin{aligned}
N&=\frac{1}{q}\sum_{(c_1,c_2)\in\mathbb{F}_q^2}\sum_{\chi}\chi\big(H(c_1,c_2)\big)\overline{\chi}(k)\\
&=q+\frac{1}{q}\sum_{\chi_0\neq\chi}\overline{\chi}(k)\sum_{(c_1,c_2)\in\mathbb{F}_q^2}\chi\big(ac_1+bc_2+c\big)\\
&=q+\frac{1}{q}\sum_{\chi_0\neq\chi}\overline{\chi}(k)\sum_{(c_1,c_2)\in\mathbb{F}_q^2}\chi_a\big(c_1\big)\chi_b\big(c_2\big)\chi(c)\\
&=q+\frac{1}{q}\sum_{\chi_0\neq\chi}\overline{\chi}(k)\chi(c)\sum_{c_1\in\mathbb{F}_q}\chi_a\big(c_1\big)\sum_{c_2\in\mathbb{F}_q}\chi_b\big(c_2\big).
\end{aligned}\tag{13.4}$$

Table 13.3 A Set of MOLSs

$L_1 =$	0	1	α	$\alpha+1$
	1	0	$\alpha+1$	α
	α	$\alpha+1$	0	1
	$\alpha+1$	α	1	0
$L_4 =$	0	1	α	$\alpha+1$
	α	$\alpha+1$	0	1
	$\alpha+1$	α	1	0
	1	0	$\alpha+1$	α
$L_5 =$	0	α	$\alpha+1$	1
	1	$\alpha+1$	α	0
	α	0	1	$\alpha+1$
	$\alpha+1$	1	0	α

As $\sum_{c\in\mathbb{F}_q}\chi_b(c)=0$ for any $b\neq 0$, note that $N=q$. Hence, $H(x,y)=ax+by+c$ is an LS if $a\cdot b\neq 0$.

Conversely, assume that the polynomial $H(x,y)=ax+by+c$ over the field $\mathbb{F}_q$ is an LS over $\mathbb{F}_q$; that is, $H(x,y)=k$ has q roots in $\mathbb{F}_q$ for any $k\in\mathbb{F}_q$. One can observe the following three possible cases:

1. If $(a,b)=(0,0)$, then $k=c$, which is a constant polynomial.
2. If exactly one of a and b is zero, then $ax+by+c=k$ becomes $ax+c=k$ or $by+c=k$. In both cases, it is a linear polynomial that has only one root.
3. If $a\cdot b\neq 0$, then $ax+by+c=k$ gives $x=\frac{1}{a}(by+c+k)$, which has q choices as y is determined by q ways. Hence, $ax+by+c=k$ has q roots only if $a\cdot b\neq 0$.

Among these three cases, only from the third case, we get q roots for the equation $H(x,y)=k$. Therefore, the polynomial $H(x,y)=ax+by+c$ is LS if and only if $a\cdot b\neq 0$ over the field $\mathbb{F}_q$. Hence, for the polynomial $H(x,y)=ax+by+c$ over the field $\mathbb{F}_q$, total $(q-1)(q-1)q=q^3-2q^2+q$ LSs exist as $a\cdot b\neq 0$ and $c\in\mathbb{F}_q$. In particular, over the field $\mathbb{F}_{2^2}$, $a\cdot b\neq 0$ and $c\in\mathbb{F}_{2^2}$, in total, 36 LSs exist.

Lemma 13.4.3. *Let $H(x,y)=a+bx^2+cy^2$ be a bivariate polynomial considered over $\mathbb{F}_q$, then it is an LS if and only if $b\cdot c\neq 0$.*

Lemma 13.4.4. *Let $H(x,y)=a+bx+cy^2$ be a bivariate polynomial considered over $\mathbb{F}_q$, then it is an LS if and only if $b\cdot c\neq 0$.*

Remark 13.4.5. *The bivariate polynomial $a+bx+cy^2$ over $\mathbb{F}_q$ is an LS over any field $\mathbb{F}_q$ if and only if $b\cdot c\neq 0$. Here, one can obtain $(q-1)^2\cdot q=q^3-2q^2+q$ LS's over $\mathbb{F}_q$.*

Lemma 13.4.6. *Let* $H(x,y) = a + bx^2 + cy$ *be a bivariate polynomial considered over* $\mathbb{F}_q$. *Then, it is an LS if and only if* $b \cdot c \neq 0$.

Remark 13.4.7. *Consider the bivariate polynomial* $a + bx^2 + cy$ *over* $\mathbb{F}_q$. *Then, it is an LS over any field* $\mathbb{F}_q$ *if and only if* $a \cdot b \neq 0$. *Here, one can obtain* $(q-1)^2 \cdot q = q^3 - 2q^2 + qLS$ *over* $\mathbb{F}_q$.

Proposition 13.4.8. *Let* $H(x,y) = a + bx + cx^2 + dy + ey^2 + fxy$ *be a bivariate polynomial considered over* $\mathbb{F}_q$. *Then, it is an LS if and only if one of the subsequent conditions is satisfied:*

1. $(b,d,f) = (0,0,0)$ *and* $c \cdot e \neq 0$.
2. $(b,e,f) = (0,0,0)$ *and* $c \cdot d \neq 0$.
3. $(c,d,f) = (0,0,0)$ *and* $b \cdot e \neq 0$.
4. $(c,e,f) = (0,0,0)$ *and* $b \cdot d \neq 0$.

In each case, $q^3 - 2q^2 + q$ *LSs of order* q *over the field* $\mathbb{F}_q$ *exist. Totally,* $4\left(q^3 - 2q^2 + q\right)LS$ *s exist.*

Proof. The first case reduces to the Lemma (13.4.3). Second is followed by the Lemma (13.4.6). The third case is followed by the Lemma (13.4.4). The fourth case is followed by Proposition (13.4.2).

Lemma 13.4.9. *The polynomial* $H(x,y) = a + bx + cx^2 + dxy$ *over* $\mathbb{F}_q$ *is not an LS for any* a,b,c,d *over* $\mathbb{F}_q$.

Proof. The proof follows from the Theorem (13.2.8), by considering $h(x) = b + cx + dy$, $f(x) = x$, and $g(y) = a$.

One can similarly prove that $H(x,y) = a + by + cy^2 + dxy$ over $\mathbb{F}_q$ is not an LS for any a,b,c,d in $\mathbb{F}_q$.

Lemma 13.4.10. *The polynomial* $H(x,y) = a + by + cy^2 + dxy$ *over* $\mathbb{F}_q$ *is not an LS for any* a,b,c,d *over* $\mathbb{F}_q$.

Proposition 13.4.11. *1. The polynomial* $H(x,y) = a + bx + cx^2 + dy + ey^2 + fxy + gx^2y$ *over* $\mathbb{F}_q$ *is not an LS if* $f \cdot g \neq 0$.

2. *The polynomial* $H(x,y) = a + bx + cx^2 + dy + ey^2 + fxy + gxy^2$ *over* $\mathbb{F}_q$ *is not an LS if* $f \cdot g \neq 0$.
3. *The polynomial* $H(x,y) = a + bx + cx^2 + dy + ey^2 + fxy + gx^2y + hxy^2$ *over* $\mathbb{F}_q$ *is not an LS if* $f \cdot g \cdot h \neq 0$.

Proof. Consider the case $1, H(x,y) = a + bx + cx^2 + dy + ey^2 + fxy + gx^2y$ over $\mathbb{F}_q$. It can be written as $H(x,y) = a + bx + cx^2 + dy + ey^2 + xy[f + gx]$. Here, it is the LS if and only if the coefficient of xy is zero, which implies $f = gx$. From Proposition (13.4.2), it is known that $H(x,y) = a + bx + cx^2 + dy + ey^2 + fxy$ forms an LS if and only if $f = 0$, which implies $g = 0$. The remaining two cases can be justified in the same line of argument.

Theorem 13.4.12. *Let* $H(x,y) = a + bx + cx^2 + dy + ey^2 + fxy + gx^2y + hxy^2 + ix^2y^2$ *be a bivariate polynomial considered over* $\mathbb{F}_q$. *Then it is LS if and only if* $g \cdot h \cdot i \neq 0$ *and one of the subsequent conditions is satisfied:*

1. $b = 0$, *either* $d = 0$ *or* $e = 0$ *but not both, and* $c \neq 0$.
2. $c = 0$, *either* $d = 0$ *or* $e = 0$ *but not both, and* $b \neq 0$.

Proof. Consider $H(x,y) = a + bx + cx^2 + dy + ey^2 + fxy + gx^2y + hxy^2 + ix^2y^2$ over $\mathbb{F}_q$. In this case, if any one of the coefficients f, g, h, i is zero, then it reduces to Proposition (13.4.11). Now, consider the first case: $b = 0, d = 0$, and $c \neq 0$. Here, the polynomial reduces to $H(x,y) = a + cx^2 + ey^2 + fxy + gx^2y + hxy^2 + ix^2y^2$. This can be written as $H(x,y) = a + bx + cx^2 + dy + ey^2 + xy[f + gx + hy + ixy]$. It forms an LS only if the coefficients of xy are zero. Hence,

$$f + gx + hy + ixy = 0.$$

This further reduces to

$$f = gx + hy + ixy.$$

Now, by substituting f in the equation, it reduces to $H(x,y) = a + cx^2 + ey^2$, which forms an LS if and only if $c \cdot e \neq 0$. So, one can obtain a total of $q^4 - 2q^3 + q^2$ LSs. The same proof follows for the remaining cases. In each case, $q^4 - 2q^3 + q^2$ LSs exist. In total, $4\left(q^4 - 2q^3 + q^2\right)$ LSs exist.

To summarize, the following observations can be made:

1. For the bivariate polynomial $ax + by$ over $\mathbb{F}_q$; it forms an LS over any field $\mathbb{F}_q$ if and only if $a \cdot b \neq 0$. Here, one can obtain a $(q-1)^2$ LSs over $\mathbb{F}_q$. Among these $(q-1)$ are symmetric LS and $(q-2)$ idempotent LS.
2. For the bivariate polynomial $ax^2 + by^2$ over $\mathbb{F}_q$, it is an LS only in the field of characteristic 2. Because the bivariate polynomial $ax^2 + by^2$ is a bivariate PP considered over $\mathbb{F}_q$ if and only if $\gcd(2, q-1) = 1$. This happens only when q is even. Thus, total $\left(2^n - 1\right)^2$ LSs over $\mathbb{F}_{2^n}$ exist.
3. In general, $ax^n + by^n$ is an LS if and only if $\gcd(n, q-1) = 1$. Here, total $(q-1)^2$ LSs over $\mathbb{F}_q$ exist.
4. It is seen that $H(x,y) = ax^2 + by^2 + cxy$ over $\mathbb{F}_q$ is not a bivariate PP considered over $\mathbb{F}_q$. One can notice that $H(x,y) = ax^2 + by^2 + cxy + dx^3$ over $\mathbb{F}_q$ is a bivariate PP, and it forms an LS over $\mathbb{F}_q$.

13.5 IDEMPOTENT AND SYMMETRIC LSS IN THE FIELD OF CHARACTERISTIC 2

In this section, the LSs constructed previously are classified into symmetric and idempotent LSs. Furthermore, enumerated symmetric and idempotent LSs of order $q = 2^n$ over the finite field $\mathbb{F}_q$.

13.5.1 Symmetric LSs in Field of Characteristic 2

The LS will be symmetric if its generating bivariate polynomial is a symmetric polynomial; that is, $H(x,y) = H(y,x)$. In this section, all the symmetric LS with their counts over the field of characteristic 2 are listed. If the polynomial is $H(x,y) = a + bx + cy$ over $\mathbb{F}_q$, then there exist $(q^2 - q)$ symmetric LSs if and only if $b = c \neq 0$ and $a \in \mathbb{F}_q$. Hence, by the definition of symmetry, $a + bx + cy = a + by + cx$, which further reduces to $(b + c)(x + y) = 0$, which is equivalent to either $x + y = 0$ or $b + c = 0$. If $x + y = 0$ implies $x = y$, then $H(x,y)$ doesn't form an LS; hence, $b + c = 0$. In other words, $b = c$, where b and c are not equal to 0. Hence, $b = c$ has $(q-1)$ choices and a is any element in $\mathbb{F}_q$. Hence, totally $(q^2 - q)$ symmetric LSs exist.

Similarly, one can see that, for the polynomial $H(x,y) = a + bx^2 + cy^2$ over $\mathbb{F}_q$, there exist $\left(q^2 - q\right)$ symmetric LSs if and only if $b = c \neq 0$ and $a \in \mathbb{F}_q$. But if $H(x,y) = a + bx + cy^2$ and $H(x,y) = a + bx^2 + cy$ are considered over $\mathbb{F}_q$, then there are no symmetric LS of this type for any a, b, c in $\mathbb{F}_q$.

Proposition 13.5.1. *Let $H(x,y) = a + bx + cx^2 + dy + ey^2 + fxy$ over $\mathbb{F}_q$. Then, there exist $2\left(q^2 - q\right)$ symmetric LS over $\mathbb{F}_q$ if either $b = d \neq 0$ and $c = e = 0$ or $b = d = 0$ and $c = e \neq 0$.*

Proof. It is known that $H(x,y) = a + bx + cx^2 + dy + ey^2 + fxy$ forms a symmetric LS over $\mathbb{F}_q$ if $H(x,y) = H(y,x)$. Hence, $a + bx + cx^2 + dy + ey^2 + fxy = a + by + cy^2 + dx + ex^2 + fxy$, and it further reduces to $bx + cx^2 + dy + ey^2 = by + cy^2 + dx + ex^2$, which is equivalent to $(b + d)x + (c + e)x^2 + (b + d)y + (c + e)y^2 = 0$ and is the polynomial mentioned in Proposition (13.4.8). Further, by Proposition (13.4.8), it forms a symmetric LS if either $b + d = 0$ and $c + e \neq 0$, or $b + d = 0$ and $c + e = 0$ or $c + e = 0$ and $b + d \neq 0$. Hence, symmetric LS exists if either $b + d = 0$ and $c + e \neq 0$ or $c + e = 0$ and $b + d \neq 0$. In both cases, $\left(q^2 - q\right)$ symmetric LSs are present, resulting in a total of $2\left(q^2 - q\right)$ symmetric LSs for the polynomial $H(x,y) = a + bx + cx^2 + dy + ey^2 + fxy$ if either $b = d \neq 0$ and $c = e = 0$ or $b = d = 0$ and $c = e \neq 0$.

Proposition 13.5.2. *Let $H(x,y) = a + bx + cx^2 + dy + ey^2 + fxy + gx^2y + hxy^2 + ix^2y^2$ over $\mathbb{F}_q$. Then, there exist $2\left(q^3 - q^2\right)$ symmetric LS over $\mathbb{F}_q$ if either $b = d \neq 0$ and $c = e = 0$ or $b = d = 0$ and $c = e \neq 0$ and $f \in \mathbb{F}_q$.*

Proof. It is known that $H(x,y) = a + bx + cx^2 + dy + ey^2 + fxy + gx^2y + hxy^2 + ix^2y^2$ over $\mathbb{F}_q$ forms a symmetric LS over $\mathbb{F}_q$ if $H(x,y) = H(y,x)$. Hence, $a + bx + cx^2 + dy$ $ey^2 + fxy + gx^2y + hxy^2 + ix^2y^2 = a + by + cy^2 + dx + ex^2 + fxy + gy^2x + hyx^2 + ix^2y^2$, and it further reduces to $bx + cx^2 + dy + ey^2 + gx^2y + hxy^2 = by + cy^2 + dx + ex^2 + gy^2x + hyx^2$, which is equivalent to $(b + d)x + (c + e)x^2 + (b + d)y + (c + e)y^2 + (g + h)$ $x^2y + (g + h)xy^2 = 0$. Here, it can be noticed that $H(x,y)$ is an LS if $f = gx + hy + ixy$, which means g, h are dependent on f. Hence, the polynomial reduces to the form mentioned in Proposition (13.4.8). Further, by Proposition (13.4.8), it forms a symmetric LS if either $b + d = 0$ and $c + e \neq 0$ or $b + d = 0$ and $c + e = 0$ or $c + e = 0$ and $b + d \neq 0$. Hence, symmetric LS exists if either $b + d = 0$ and $c + e \neq 0$ or $c + e = 0$ and $b + d \neq 0$. In both

cases, $\left(q^3-q^2\right)$ symmetric LSs are present, resulting in a total of $2\left(q^3-q^2\right)$ symmetric LSs existing for the polynomial $H(x,y)=a+bx+cx^2+dy+ey^2+fxy+gx^2y+hxy^2+ix^2y^2$ over $\mathbb{F}_q$ if either $b=d\neq 0$ and $c=e=0$ or $b=d=0$ and $c=e\neq 0$ and $f\in\mathbb{F}_q$.

13.5.2 Idempotent LSs in Field of Characteristic 2

A bivariate PP $H(x,y)$ is idempotent if $H(x,x)=x$. The corresponding LS will be idempotent if its generating bivariate polynomial is an idempotent polynomial. In this section, all the idempotent LSs and their counts in the field of characteristic 2 are presented. For the polynomial $H(x,y)=a+bx+cy$ over $\mathbb{F}_q$, there exist $(q-2)$ idempotent LSs if and only if $b+c=1$ for any $b,c\in\mathbb{F}_q^*$. It can be verified by assuming that $H(x,y)=a+bx+cy$ is the LS of order q over $\mathbb{F}_q$ as follows. It is known that $H(x,y)=a+bx+cy$ forms an idempotent LS over $\mathbb{F}_q$ if $H(x,x)=x$. Hence, at (0, 0) the polynomial reduces to $a=0$, which is equivalent to $H(x,y)=bx+cy=0$. Further, by the definition of idempotent, it can be simplified as $x(b+c)=x$, which implies that $b+c=1$, where the values of b and c are other than 0 and 1. Hence, the $b=c$ has $(q-2)$ choices. But for the polynomial $H(x,y)=a+bx^2+cy^2$ over $\mathbb{F}_q$, there are no idempotent LS of this type for any a,b,c in $\mathbb{F}_q$. As in the previous case, on simplification, it reduces to $x^2(b+c)=x$, which implies that either $x(b+c)=1$ or $x=0$, where the values of b and c are other than 0. Hence, in both cases, LSs don't exist. So, there does not exist an idempotent LS for the polynomial $H(x,y)=a+bx^2+cy^2$ over $\mathbb{F}_q$. Similarly, for $H(x,y)=a+bx+cy^2$ and $H(x,y)=a+bx^2+cy$ over $\mathbb{F}_q$, there are no idempotent LS for any a,b,c in $\mathbb{F}_q$.

Proposition 13.5.3. *Let $H(x,y)=a+bx+cx^2+dy+ey^2+fxy$ over $\mathbb{F}_q$. Then, there exist $(q-2)$ idempotent LS over $\mathbb{F}_q$.*

Proof. It is known that $H(x,y)=a+bx+cx^2+dy+ey^2+fxy$ is an idempotent LS over $\mathbb{F}_q$ if $H(x,x)=x$. Hence, at (0, 0) the polynomial reduces to $a=0$, which is equivalent to $H(x,y)=bx+cx^2+dy+ey^2+fxy=0$. Further, by the definition of idempotent, it reduces to $bx+cx^2+dx+ex^2+fx^2=x$, which is equivalent to $(b+d)+(c+e+f)x=1$. By Proposition (13.4.8), it reduces to four cases; in the first three cases of Proposition (13.4.8), an idempotent LS does not exist. In the fourth case, $(q-2)$ idempotent LSs exist only if $b+d=1$ and $(c,e,f)=(0,0,0)$.

Proposition 13.5.4. *Let $H(x,y)=a+bx+cx^2+dy+ey^2+fxy+gx^2y+hxy^2+ix^2y^2$ over $\mathbb{F}_q$. Then, there exists $(q-2)$ idempotent LS over $\mathbb{F}_q$ if $b+d=1$ and $(c,e,f,g,h,i)=(0,0,0,0,0,0)$.*

Proof. It is known that $H(x,y)=a+bx+cx^2+dy+ey^2+fxy+gx^2y+hxy^2+ix^2y^2$ is an idempotent LS over $\mathbb{F}_q$ if $H(x,x)=x$. Hence, at (0,0) the polynomial reduces to $a=0$, which is equivalent to $H(x,y)=bx+cx^2+dy+ey^2+fxy+gx^2y+hxy^2+ix^2y^2=0$. Further, by the definition of idempotent, it reduces to $bx+cx^2+dx+ex^2+fx^2+gx^3+hx^3+ix^4=x$, which is equivalent to $(b+d)+(c+e+f)x+(g+h)x^2+ix^3=1$. Here, assume that $H(x,y)$ is an LS, so $f=gx+hy+ixy$. Substituting the value of f in the previous expression, it reduces to $(b+d)+(c+e)x=1$. By Theorem (13.4.12), it reduces to four cases; in

the first three cases of Theorem (13.4.12), an idempotent LS does not exist. In the fourth case, $(q-2)$ idempotent LSs exist only if $b+d=1$ and $(c,e,f,g,h,i)=(0,0,0,0,0,0)$.

13.6 MUTUALLY ORTHOGONAL AND SOLSS

From a given set of LSs, one can anticipate the construction or identification of MOLSs. Using pairs of orthogonal Latin squares (OLSs), various kinds of encryption schemes can be generated. Dougherty et al. (2024), presented an image encryption algorithm using a pair of MOLSs L_1 and L_2. And if one applies row and column permutations to a pair of OLSs, they remain OLSs. Using these two OLSs, one can construct a magic square M by L_1+nL_2 of order n. Moreover, encryption and decryption of messages can be done using a pair of OLSs in two different ways. One way is if two LSs, L_1 and L_2, are considered with a message (a,b). Assume two parties agree on the common keys L_1 and L_2. Now, the sender checks a position of entry a in L_1 and b in L_2, and then the corresponding position (i,j) is a ciphertext for the plaintext (a,b). The receiver will decrypt it by verifying the position (i,j) in L_1 and L_2. Another way is that the sender can consider the message (a,b) as a position in L_1 and L_2 and can send the entries of the position (a,b) as a ciphertext.

Let us consider the example with $H(x,y)=ax+by$ over the field $\mathbb{F}_{2^2}=\{0,1,\alpha,\alpha+1\}$. One gets a list of nine different combinations of $a,b\in\mathbb{F}_{2^2}$ for which $H(x,y)$ forms an LS. Among these nine LSs, superpositions of any two L_i give a total of 54 pairs of OLSs. By considering the OLSs L_4 and L_5, which are mentioned in Example (13.4.1), if the considered message is $(\alpha,1+\alpha)$, then $(0,\alpha)$ will be the ciphertext. Similarly, for the second method, if the message is $(\alpha,1+\alpha)$, the ciphertext will be $(0,1+\alpha)$. As it is known, using the bivariate PPs, one can get a pair of OLSs of order 2^n in the finite field $\mathbb{F}_{2^n}$. Instead of sending an entire pair of OLS sets, one can use a generalized bivariate PP and just agree upon the coefficient as a key for the entire communication.

For example, if the considered polynomial is $H(x,y)=a+bx+cy^2$ over $\mathbb{F}_{2^2}$, a total of 864 pairs of OLSs exist for all values of $a,b,c\in\mathbb{F}_{2^2}$, except in the following cases:

1. Each $b\in\mathbb{F}_{2^2}^*$ such that $(a,b,b)=(a,c,c),\forall c\in\mathbb{F}_{2^2}^*$ and $a\in\mathbb{F}_{2^2}$.
2. Each $b,c\in\mathbb{F}_{2^2}^*$ and $b\neq c$ such that
   i) (a,b,c) and $(a,b+c,b)$, ii) (a,c,b) and $(a,c,b+c)$, $\forall a\in\mathbb{F}_{2^2}$.
3. Each $a,b\in\mathbb{F}_{2^2}^*$ and $a\neq b$ such that $(a,b,c)=(a,b,c)$ and $(a,c,b)=(a,c,b),\forall a\in\mathbb{F}_{2^2}^*$.

So one can use other coefficients than these as a key for encryption and decryption. The same exception applies if any other polynomials mentioned in the previous section are considered. Moreover, a pair of OLSs can be constructed in the same manner using the general bivariate polynomial (1).

An LS is self-orthogonal if it is orthogonal to its transpose. When compared with ordinary LS, SOLS has some particular properties. Moreover, it reduces the resources required for design compared to a pair of OLSs. Recently, Xu and Tian (2018) proposed a novel image encryption algorithm based on SOLS. Authors adopted a permutation–substitution–permutation architecture in one round of encryption and used one SOLS to perform the whole encryption process. Meanwhile, SOLS can be identified using the bivariate PPs. In this case, just the

coefficients can be used as the key instead of sending the LS and its transpose. For the polynomial (1), it is observed that if the polynomial is homogeneous, then only the list of SOLS exists, and otherwise, it doesn't. For example, if the polynomial is $H(x,y) = a + bx^2 + cy^2$ over $\mathbb{F}_{2^2}$, a total of 24 pairs of SOLS exist among 36 for any values of $b, c \in \mathbb{F}_{2^2}^*$ such that $b \neq c, \forall a \in \mathbb{F}_{2^2}$. But for the polynomial $H(x,y) = a + bx + cy^2$, there does not exist any SOLS over $\mathbb{F}_{2^2}$.

13.7 CONCLUSION

Within this chapter, a bivariate PP, as defined in Equation (13.1), is identified over the finite field of characteristic 2. This polynomial is shown to represent an LS of order 2^n. Furthermore, the LS of order 2^n for each mentioned bivariate PP with certain conditions on the coefficients over the finite field of characteristic 2 is characterized and enumerated. In addition, the existence and enumeration of symmetric, idempotent, mutually orthogonal, and SOLS for the bivariate PP mentioned in Equation (13.1) over the finite field of characteristic 2 are investigated. It is clear to see that every LS of order 4 is covered by the polynomial stated in Equation (13.1).

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Chapter 14

PQLS-Evote

A Novel Post-Quantum Secure and Efficient Lattice-Based Signcryption Scheme for Electronic Voting

Sourav and Rifaqat Ali

14.1 INTRODUCTION

Voting stands as a vital instrument for democratic expression, facilitating collective decision-making and empowering citizens to influence policies, leadership, and governance. Since the emergence of modern democratic systems, traditional paper-based voting has played a central role due to its perceived transparency, simplicity, and auditability. However, as populations expand and electoral procedures become more complex, conventional methods face mounting challenges such as logistical inefficiencies, human error, voter fraud, and difficulties in ensuring accessibility and scalability (Mursi et al. 2013; Singh et al. 2023b; Ali 2025).

The digital transformation of civic infrastructure has brought forward electronic voting (e-voting) systems as a promising evolution of conventional balloting. E-voting provides advantages such as faster result computation, lower operational costs, accessibility for remote or disabled voters, and automation of auditing processes. Despite these benefits, the shift to digital platforms introduces critical concerns related to trust, security, and transparency. Specifically, e-voting systems must ensure that each vote is cast as intended, recorded accurately, and counted reliably, all while protecting against both internal and external threats (Ohize et al. 2025). Ensuring voter privacy, resistance to tampering, and verifiability of results remains a complex and active research problem.

To address these security needs, cryptographic techniques are integral to the e-voting architecture. Classical cryptographic systems, including Rivest–Shamir–Adleman (RSA), digital signature algorithm (DSA), and elliptic curve cryptography (ECC), have historically underpinned secure transmission and authentication (Singh et al. 2023a; Sharma et al. 2024a, 2024b; Kowsalya et al. 2021). These systems facilitate core requirements such as data confidentiality, digital signatures, secure authentication, and voter anonymity (Ali and Pal 2018). However, the long-term security of these schemes is increasingly uncertain due to the anticipated impact of quantum computing.

Quantum computing poses a paradigm shift in computational power, capable of breaking cryptographic primitives that rely on the intractability of integer factorization and discrete logarithms. Shor's algorithm, for instance, can efficiently solve such problems, rendering RSA and ECC insecure in a post-quantum future (Mansoor et al. 2025; Singh et al. 2020). As scalable quantum devices approach feasibility, the integrity, privacy, and trustworthiness of existing e-voting systems are at substantial risk (Qu et al. 2022). This urgent threat has catalyzed global interest in post-quantum cryptography (PQC), a new wave of cryptographic design secure against both classical and quantum adversaries (Bernstein and Lange 2017; Sourav and Ali 2024; Ali and Obaidat 2025b).

Among PQC candidates, lattice-based cryptography has emerged as a leading contender due to its robustness, efficiency, and firm mathematical grounding. Lattice-based schemes rely on hard problems such as learning with errors (LWE) and short integer solution (SIS), which remain secure against known quantum algorithms (Choudhary and Ali 2024; Singh et al. 2025). These

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schemes are also well-suited for practical deployment due to their polynomial-time performance, small key sizes, and flexible construction of advanced primitives such as identity-based encryption, group signatures, and signcryption.

In this context, signcryption—a cryptographic primitive that simultaneously performs digital signing and encryption—offers significant advantages for e-voting applications. By merging two operations into a single, efficient process, signcryption reduces computational cost, bandwidth consumption, and communication overhead, all of which are critical for large-scale elections (Ullah et al. 2021; Sourav and Ali 2025a). Moreover, the ability to guarantee confidentiality and authenticity in one operation streamlines implementation and simplifies cryptographic workflows.

Another critical component in the design of trustworthy e-voting systems is threshold cryptography. A (t, n) threshold scheme enables decentralized trust by distributing cryptographic control across multiple authorities. This approach ensures that no single entity can compromise the system, and the protocol remains resilient even if up to $(n - t)$ parties are corrupted (Shamir 1979; Kumar and Padhye 2023; Liu and Zhao 2019). Threshold cryptography supports fault tolerance, secure key management, and system robustness—particularly relevant for adversarial environments like national elections.

Problem definition: Despite significant advances, there remains a lack of unified cryptographic solutions that offer post-quantum security, efficiency, and robust functionality (such as signcryption and thresholding) tailored specifically for e-voting systems. Most existing approaches either suffer from performance limitations or lack formal post-quantum guarantees. To bridge this gap, this chapter presents **PQLS-EVote:** a novel post-quantum lattice-based signcryption scheme for electronic voting. This work contributes a unified cryptographic framework that meets the stringent demands of next-generation e-voting infrastructure. The scheme is designed using hard lattice problems (SIS and LWE), integrates a (t, n) threshold mechanism for decentralized trust, and applies rejection sampling to achieve efficient signature generation with minimized overhead.

In conclusion, the proposed PQLS-EVote framework represents a comprehensive solution that addresses the core security, efficiency, and trust challenges of electronic voting in the quantum era. By combining signcryption, lattice-based constructions, and threshold cryptography, this work advances the field of secure e-voting and sets a foundation for scalable, future-proof democratic participation systems.

14.1.1 Post-Quantum Cryptography

PQC refers to cryptographic systems designed to remain secure even in the presence of quantum computers. Quantum computing, which harnesses principles of quantum mechanics to perform computations, poses a serious threat to traditional cryptographic schemes such as RSA, ECC, and DSA. These classical systems depend on problems like integer factorization and the discrete logarithm problem, both of which can be solved efficiently on a quantum computer using Shor's algorithm (Shor 1999).

To counter this threat, the cryptographic community has been actively developing algorithms based on problems conjectured to be hard even for quantum computers. These approaches form the foundation of PQC, aiming to ensure long-term confidentiality, integrity, and authenticity of sensitive data in the post-quantum era (Bernstein and Lange 2017; Upadhyay and Padhye 2025). The main families of post-quantum cryptographic schemes include the following:

- **Lattice-Based Cryptography:** Utilizes hard problems such as LWE and the shortest vector problem (SVP). These schemes are efficient and highly versatile, supporting functionalities like encryption, digital signatures, and fully homomorphic encryption (FHE) (Ajtai 1996; Lyubashevsky 2012).

- **Code-Based Cryptography:** Based on the difficulty of decoding linear error-correcting codes. The McEliece cryptosystem, one of the earliest examples, remains unbroken since its inception in 1978 (Bernstein et al. 2008; Kumari et al. 2022).
- **Multivariate Polynomial Cryptography:** Uses the intractability of solving systems of multivariate quadratic equations. Notable schemes include Rainbow and Unbalanced Oil and Vinegar (Kipnis and Shamir 1999; Omar et al. 2023).
- **Hash-Based Cryptography:** Leverages secure cryptographic hash functions to construct digital signatures with minimal assumptions. Examples include the Merkle Signature Scheme and the Leighton–Micali Signature (LMS) (McGrew et al. 2019).
- **Isogeny-Based Cryptography:** Employs hard problems related to isogenies between supersingular elliptic curves. Though offering compact key sizes, these schemes tend to be computationally expensive. The Supersingular Isogeny Key Encapsulation is a well-known protocol in this domain (Jao and De Feo 2011).

Each family presents unique trade-offs in terms of computational efficiency, key and ciphertext sizes, and implementation complexity. To facilitate widespread adoption, the National Institute of Standards and Technology (NIST) is standardizing several PQC algorithms through a rigorous multi-round evaluation process.

14.1.1.1 Mathematical Background

For instance, classical RSA security relies on the hardness of factoring:

$$\text{Given } N = pq, \text{ find } p \text{ and } q.$$

Quantum computers, using Shor's algorithm, can solve this in polynomial time. In contrast, PQC schemes are built on problems like lattice-based or hash-based constructions that remain intractable even for quantum adversaries.

14.1.1.2 Hash-Based Signature Example—LMS

The LMS scheme employs Merkle hash trees for secure digital signatures:

- Generate a Merkle tree using one-time signature (OTS) keys
- Sign messages with OTS keys
- Verify using the Merkle root and authentication path

Let M be a message and K a one-time private key. The signature is:

$$\sigma = \text{Hash}(K \parallel M)$$

The verifier checks σ using the Merkle root.

14.1.2 Lattice-Based Cryptography

Lattice-based cryptography is one of the most promising and actively researched domains within PQC. It derives its security from hard mathematical problems defined over lattices—regular grids of points in n-dimensional space:

$$\mathcal{L} = \left\{ \sum_{i=1}^{n} z_i \mathbf{b}_i \,\middle|\, z_i \in \mathbb{Z} \right\}$$

where $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n \in \mathbb{R}^n$ are linearly independent basis vectors.

14.1.2.1 Core Hard Problems

1. **SVP:** Find the shortest non-zero vector in $\mathcal{L}$.
2. **LWE:** Given $(\mathbf{A}, \mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e})$, recover the secret vector $\mathbf{s}$, where $\mathbf{e}$ is a small error (Ajtai 1996).

These problems are believed to be resistant even to quantum attacks, making them ideal for post-quantum cryptographic protocols.

14.1.2.2 LWE-Based Encryption Scheme (Simplified)

- **Key Generation:** Select a secret vector $\mathbf{s} \in \mathbb{Z}_q^n$, a public matrix $\mathbf{A} \in \mathbb{Z}_q^{m \times n}$, and an error vector $\mathbf{e}$.
- **Encryption:** For message $m \in \{0,1\}$, generate ciphertext $(\mathbf{u}, v)$:

$$\mathbf{u} = \mathbf{A}^T \mathbf{r}, \quad v = \mathbf{b}^T \mathbf{r} + \lfloor q/2 \rfloor m$$

- **Decryption:** Compute $v - \mathbf{u}^T \mathbf{s}$ and round to retrieve m.

Example: Let $q = 17, \mathbf{s} = [1,2], \mathbf{A} = \begin{bmatrix} 3 & 5 \\ 7 & 1 \end{bmatrix}$, and $\mathbf{e} = [1,1]$. Then:

$$\mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e} = [14,10]$$

This tuple $(\mathbf{A}, \mathbf{b})$ forms an LWE sample, used for encryption.

14.1.2.3 Applications

Lattice-based cryptography supports diverse functionalities:

- **Public Key Encryption:** for example, Kyber (NIST finalist)
- **Digital Signatures:** for example, Dilithium (NIST standard)
- **FHE:** Enables computations over encrypted data (Ducas et al. 2013).

14.1.2.4 Use Case in E-Voting

Lattice-based encryption ensures:

- Voter privacy and ballot secrecy
- Authentication of eligible voters
- Threshold decryption for tallying without exposing individual votes

Threshold decryption using lattice trapdoors enables secure multiparty computation, vital in scenarios like secure elections or collaborative decryption in healthcare data sharing.

Lattice-based cryptography stands at the forefront of post-quantum secure solutions. With robust security assumptions, quantum resistance, and the ability to support advanced features like FHE, it is well-positioned to become a foundational technology in the quantum

era. Its growing adoption in applications like e-voting, secure messaging, and digital identity management underscores its transformative potential (Peikert 2016).

14.1.3 Need for Cryptography in Electronic Voting Systems

Electronic voting (e-voting) systems are becoming increasingly relevant in modern democracies due to their potential to streamline the voting process, increase accessibility, and improve efficiency in election management. However, the digital nature of e-voting introduces significant security and privacy concerns, making cryptography an essential component in its design and implementation.

Cryptography plays a vital role in ensuring several critical properties in electronic voting systems:

- **Confidentiality:** Voters' choices must remain secret to preserve privacy and prevent coercion.
- **Integrity:** Votes must not be altered, deleted, or fabricated, ensuring that election results reflect the actual choices of the voters.
- **Authenticity:** Only eligible voters should be able to cast a vote, and each voter must be allowed to vote only once.
- **Verifiability:** Voters and auditors must be able to verify that all votes were correctly counted without compromising individual privacy.
- **Anonymity:** It must be computationally infeasible to link a vote to the voter who cast it, ensuring resistance to vote tracing and retribution.

To meet these requirements, traditional cryptographic primitives such as public key encryption, digital signatures, homomorphic encryption, and zero-knowledge proofs have been widely used in e-voting systems. However, with the advent of quantum computing, many of these classical cryptographic schemes are at risk. Quantum algorithms such as Shor's algorithm can efficiently break RSA and ECC, undermining the confidentiality, authenticity, and integrity mechanisms currently deployed in e-voting infrastructure (Shor 1999).

Therefore, there is an urgent need to adopt post-quantum cryptographic solutions in electronic voting systems. Quantum-resistant algorithms, particularly those based on lattice problems, offer a promising way forward. They provide strong security guarantees even in the presence of quantum adversaries, ensuring the long-term confidentiality and trustworthiness of electronic voting systems.

This study aims to design a lattice-based signcryption scheme tailored for e-voting systems, integrating post-quantum cryptographic principles and threshold cryptography to enhance both security and efficiency. By addressing the unique challenges posed by quantum computing, the proposed scheme aims to ensure the confidentiality, authenticity, anonymity, and verifiability of electronic votes in the face of future technological advancements.

Contributions of the Chapter

Motivated by the urgent need for quantum-resistant and efficient e-voting mechanisms, this work introduces a novel signcryption scheme grounded in lattice-based cryptography and augmented with a threshold secret-sharing mechanism. The proposed scheme is designed to meet the stringent security requirements of electronic voting—ensuring confidentiality, authenticity, anonymity, and verifiability—while maintaining computational and

communication efficiency suitable for real-world deployment. The major contributions of this paper are as follows:

- The proposed scheme is based on the hardness of LWE and SIS problems, offering strong resistance against quantum adversaries.
- The scheme utilizes rejection sampling techniques to optimize ring signature generation and reduce performance overhead.
- A (t, n) threshold mechanism enhances the decentralization and robustness of the system, ensuring fault tolerance and secure key management.
- A rigorous security analysis is conducted in the random oracle model (ROM), demonstrating resistance to various attack vectors, including chosen ciphertext and forgery attacks.
- Comparative analysis shows that the proposed scheme outperforms existing post-quantum and classical alternatives in both security and efficiency metrics.

Organization of the Chapter

The rest of the chapter is organized as follows. In Section 14.2, we explore the existing body of literature related to our work. Section 14.3 lists basic tools to understand lattice-based cryptography. Section 14.4 provides a detailed explanation of our proposed scheme, PQLS-EVote. In Section 14.5, we demonstrate the correctness of the protocol. The security analysis is covered in Section 14.6. Section 14.7 presents the performance evaluation of the proposed protocol. Section 14.8 summarizes the results and discussions of the proposed work, and, finally, section 14.9 concludes the chapter with future scope.

14.2 RELATED WORKS

Electronic voting (e-voting) systems have evolved significantly, leveraging various cryptographic techniques to ensure security, privacy, and integrity. Mursi et al. (2013) provide a detailed survey on the evolution of electronic voting systems, outlining key functional and constitutional requirements such as privacy, transparency, and incoercibility. The chapter classifies existing e-voting schemes, analyzes their strengths and weaknesses, and offers suggestions for improving recent systems. Traditional schemes often employ homomorphic encryption, blind signatures, and mix-nets. Blind signature schemes ensure voter anonymity by allowing a message to be signed without revealing its content, as demonstrated in several e-voting protocols (Aziz 2019). Mix-net-based approaches shuffle encrypted votes to obscure the link between voters and their ballots, enhancing privacy (Fujioka et al. 1993). Liu and Zhao (2019) introduced a novel e-voting scheme that combined secret sharing and k-anonymity to enhance security and privacy. The proposed system ensures fundamental properties such as confidentiality and verifiability, while also achieving coercion-resistance and unconditional security without relying on computational assumptions.

The integration of blockchain technology into e-voting systems has garnered attention due to its decentralized and immutable nature. Blockchain ensures transparency and tamper-resistance, critical for maintaining trust in electoral processes. Mukherjee et al. (2023) proposed a privacy-preserving blockchain-based e-voting system that leverages smart contracts to automate and secure the voting process. Similarly, Spanos and Kantzavelou (2023) introduced EtherVote, a blockchain-based system emphasizing voter authentication and data integrity. The proposed systems aimed to address issues like vote manipulation and unauthorized access.

The advent of quantum computing poses significant threats to classical cryptographic algorithms, necessitating the exploration of PQC solutions. Lattice-based cryptography has emerged as a promising PQC approach due to its resistance to quantum attacks. Boyen et al. (2020) proposed a practical, verifiable lattice-based decryption mix-net, ensuring vote confidentiality and integrity. These schemes rely on hard lattice problems like LWE and SIS to provide security against quantum adversaries. Zhou et al. (2022) introduced a ring signature scheme based on the lattice and its application to anonymous electronic voting, utilizing the hard mathematical problem: SIS problem.

Signcryption, which combines signing and encryption in a single step, offers efficiency and security benefits in e-voting systems. Qu et al. (2022) introduced an e-voting protocol based on blockchain and homomorphic signcryption, enhancing both confidentiality and authenticity. Threshold cryptography, particularly Shamir's (t,n) threshold scheme (Shamir 1979), allows a secret to be distributed among multiple parties, requiring a minimum number of participants to reconstruct it. This approach enhances fault tolerance and security in e-voting by preventing single points of failure. (Sourav and Ali 2024), designed a lattice-based ring signcryption scheme that utilizes the (t,n) threshold scheme for decentralized storage of patient health records, aimed at smart healthcare management. Abed et al. (2024) proposed a blockchain-based electronic voting system aimed at addressing key issues of centralization, voter anonymity, and result confidentiality. Authors highlighted how decentralization through blockchain can enhance trust in the electoral process while mitigating concerns related to manipulation and security breaches. Ohize et al. (2025) presented a comprehensive survey on the integration of blockchain technology into electronic voting (e-voting) systems, highlighting recent architectures, global trends, and emerging solutions. It addresses key challenges such as security, scalability, and auditability, while outlining future research directions to enhance trust and efficiency in blockchain-based e-voting.

The literature indicates a clear trend toward integrating advanced cryptographic techniques, such as lattice-based cryptography and blockchain technology, into e-voting systems to address emerging security challenges, particularly those posed by quantum computing. Signcryption and threshold cryptography further contribute to the development of secure and efficient e-voting protocols. The proposed scheme, PQLS-EVote, builds upon these advancements, aiming to provide a post-quantum secure and efficient signcryption scheme for electronic voting, leveraging lattice-based cryptography.

14.3 PRELIMINARIES

This section outlines essential background concepts related to the proposed cryptographic approach. For clarity and consistency, Table 14.1 summarizes the key notations that are used throughout the chapter.

14.3.1 Key Definitions

This section introduces fundamental definitions that are frequently used in the context of lattice-based cryptography.

Definition 14.1. *Lattice-based cryptography: It is a type of cryptographic construction that relies on the mathematical structures known as lattices. Lattices are sets of points in a multi-dimensional space with certain mathematical properties. In lattice-based cryptography, the hardness of certain problems related to lattices forms the basis for building cryptographic primitives and protocols. Lattice-based cryptography is considered*

Table 14.1 List of Notations with Descriptions

Notation	*Description*
$\mathbb{Z}$	Set of integers
$\mathbb{R}$	Set of real numbers
q	Prime number
$\mathbb{Z}_q^{m\times n}$	$m \times n$ matrices set with integers modulo q
D_σ^m	Discrete Gaussian distribution
$\rho_{\sigma,c}$	Gaussian function
χ	Probability distribution
$f(x)$	Polynomial in x
$\omega()$	Space complexity function
$O()$	Time complexity function
$H(.)$	Hash function
$\oplus$	XOR operation
L	Lattice structure
I_i	Identity of ith entity
X_i	Secret key of ith member
Y_i	Public key of ith member
ς, ζ	Hash values of $H_3 : \{0,1\}^* \rightarrow \{0,1\}^d$
M	Message
C	Ciphertext
s	Sender
r	Receiver
l	Number of ring members
n	Security parameter
$\mathcal{A}, \mathfrak{C}$	Adversary and challenger
U	Set of ring members
V	Voter
EC	Election Commission
EA	Election Authority
CA	Counting Authority
MS	Main Server of Election Authority

post-quantum secure, meaning that it is believed to be resistant to attacks by quantum computers (Micciancio and Regev 2009).

Definition 14.2. *Lattice: An n-dimensional lattice L over $\mathbb{Z}^n$ is a discrete additive subgroup formed by linear combinations of linearly independent vectors $x_1, x_2, \ldots, x_n \in \mathbb{Z}^n$ with integer coefficients. It can be defined as:*

$$L(x_1, x_2, \ldots, x_n) = \left\{ \sum_{i=1}^{n} \alpha_i x_i \,\middle|\, \alpha_i \in \mathbb{Z} \right\}$$

where $B = \{x_1, x_2, \ldots, x_n\}$ *represents a basis for the lattice. The dual or orthogonal lattice associated with* B *is given by:*

$$L^{\perp}(B) = \{ x \in \mathbb{Z}^m \mid Bx \equiv 0 \pmod{q} \}$$

as introduced in Ajtai (1999).

Geometrically, a lattice consists of regularly spaced points extending throughout n-dimensional space in a repeating pattern.

Definition 14.3. *SIS problem: Given a modulus* q, *a positive real number* β, *a matrix* $A \in \mathbb{Z}_q^{n \times m}$ *whose columns are* m *uniformly random vectors* $\alpha_i \in \mathbb{Z}_q^n$, *the goal is to find an integer vector* $x \in \mathbb{Z}_q^n$ *such that* $Ax = \sum_i \alpha_i x_i = 0 \bmod q$, *where* $\|x\| \leq \beta$ *(Ajtai 1999).*

Definition 14.4. *Ring LWE (RLWE) problem: Let* R_q *denote a polynomial ring modulo a prime* q. *The RLWE problem is defined as the task of recovering a secret element* $\mathrm{s} \in R_q$, *given access to a set of samples of the form* $(\mathrm{a}, t = \mathrm{as} + e)$, *where* $\mathrm{a} \in R_q$ *is chosen uniformly at random,* e *is an error term sampled from a noise distribution* χ *over* R_q, *and* t *is the resulting output. The problem is parameterized by integers* n *and* q, *with* χ *denoting the chosen error distribution. This formulation was introduced in Regev (2004).*

Definition 14.5. *Decision Ring LWE (D-RLWE) Problem: The goal is to distinguish whether a given* (a, t) *pair originates from the RLWE distribution* $A_{\mathrm{s},\chi}$, *defined by* $t = \mathrm{as} + e$ *with secret* $\mathrm{s} \in R_{,}$ *and error* e *sampled from a noise distribution* χ *or whether the pair is uniformly chosen at random from* $R_q \times R_q$. *This decision-making task assumes the following conditions:* n, k, *and* q *(where* $q = 2^k$*) are positive integers,* χ *represents the distribution over the ring* R_q, e *follows an error distribution, and* $\mathrm{a} \,\&\, \mathrm{s}$ *are randomly selected from the ring* R_q *(Regev 2004).*

In lattice-based cryptographic schemes, Gaussian distributions are commonly adopted due to their efficiency and suitability.

Definition 14.6. *Gaussian function: Given* $\sigma \in Z_q^m$ *the standard deviation and vector* $v \in Z_q^m$ *is the center, the Gaussian function is defined as* $\rho_{\sigma,v} = e^{\frac{-\pi^2 \|x-c\|^2}{2\sigma^2}}$ *(Ma et al. 2019).*

Definition 14.7. *Gaussian distribution: For any* $v \in \mathbb{Z}_q^m$ *and Gaussian parameter* σ, *the discrete Gaussian distribution over* m*-dimensional lattice* L *is defined as:* $\forall x \in L, D_{\sigma,v}^m(x) = \frac{\rho_{\sigma,v}(x)}{\rho_{\sigma,v}(L)}$, *where* $\rho_{\sigma,v}(x)$ *denotes the Gaussian function centered at* v *(Ma et al. 2019).*

Definition 14.8. *Signcryption: Signcryption is a cryptographic method that combines the processes of encryption and signature into a single, efficient operation. A typical signcryption scheme involves four main algorithms:*

- *Setup: This algorithm initializes the system by taking a security parameter as input and generating the global parameters necessary for the cryptographic operations.*
- *Key Generation: This process generates key pairs, typically consisting of a public key and a corresponding private key, using the system parameters and the identity of the user.*
- *Signcryption: This algorithm simultaneously signs and encrypts a message. It takes the global system parameters, the receiver's public key, the sender's private key,*

and the message to be transmitted, producing a ciphertext that is both signed and encrypted.

- *Unsigncryption: The unsigncryption algorithm is responsible for decrypting the signcrypted message. It uses the receiver's private key, the sender's public key, and the system parameters to recover the original message, or it indicates failure if the process cannot be completed successfully.*

14.3.2 Lattice Sampling Algorithms

The following are some probabilistic polynomial-time algorithms which we used in our proposed scheme:

- **TrapGen** (n,m,q) (Yu and Lv 2023): A security parameter n, prime q = *poly (n)*, and $m \geq c\ n \log q$, where c is a fixed constant are input, the **TrapGen** (n,m,q) algorithm gives the matrix $A \in \mathbb{Z}_q^{n\times m}$ and base B_A of lattice $L^{\perp}(A) = \{z \in \mathbb{Z}^m : Az = 0 \bmod q\}$, where $\|B_A\| \leq O(n \log q)$.
- **BasisDel** (A,D,B_A,σ) (Yu and Lv 2023): Given a matrix $A \in \mathbb{Z}_q^{n\times m}$, an invertible square matrix D, B_A be a basis of $L^{\perp}(A)$, a Gaussian perimeter $\sigma \geq \|\overline{B_A}\|\omega(\sqrt{log\ n})$ and $R = AD^{-1}$. As a result, this algorithm yields a basis $\mathcal{B}$ for the orthogonal lattice $L^{\perp}(R)$.
- **SampleMatPre** $(R,\mathcal{B},\sigma,D)$ (Yu and Lv 2023): Given a matrix $R \in \mathbb{Z}_q^{n\times m}$ and a set base $\mathcal{B}$ of $L^{\perp}(R)$ and a Gaussian parameter $\sigma \geq \|\overline{B_A}\|\ \omega(\sqrt{log\ n})$, the **SampleMatPre** $(R,\mathcal{B},\sigma,D)$ algorithm gives a matrix $M \in \mathbb{Z}_q^{m\times b}$ satisfying $RM = D$, with $\|M\| \leq \sigma\sqrt{m}$ and $M \in \mathbb{Z}_q^{m\times b}$.

14.3.3 (t,n) Threshold Scheme

The (t,n) threshold scheme (Shamir 1979; Huang and Chang 2006) consists of two phases: setup and recovery.

Setup: The sender, denoted by $\mathcal{S}$, intends to distribute a secret value $S \geq 0$ among n users.

- $\mathcal{S}$ selects a prime number $p > \max(S,n)$ and sets $a_0 = S$.
- Then, $\mathcal{S}$ chooses $t-1$ random and independent coefficients $a_1, a_2, \ldots, a_{t-1}$ to construct a polynomial of degree $t-1$:

$$f(x) = \sum_{i=0}^{t-1} a_i x^i.$$

- For each user of respective identifier $\mathcal{U}_i$ with index $i \in \{1,2,\ldots,n\}$, the sender computes the share $S_i = f(\mathcal{U}_i) \bmod p$ and securely transmits the pair $(\mathcal{U}_i, S_i)$ to the corresponding user.

Recovery: Any subset of at least t users can collaboratively reconstruct the original secret S using their individual shares and the Lagrange interpolation method (Leevik et al. 2023).

Each participating user, identified by a unique identifier $\mathcal{U}_i$, computes a Lagrange coefficient as follows:

$$l_i = (-1)^{t-1} \prod_{\substack{j \neq i \\ 1 \leq j \leq t}} \frac{\mathcal{U}_j}{\mathcal{U}_i - \mathcal{U}_j}. \tag{14.1}$$

Using the computed coefficients and corresponding share values S_i, the secret is recovered by reconstructing the polynomial:

$$f(0) = \sum_{1 \leq i \leq t} S_i \cdot l_i. \tag{14.2}$$

The constant term $f(0)$ gives the original secret $S = a_0$, successfully completing the recovery phase.

14.3.4 Security Model

This section outlines the security model as an interactive game between an adversary $\mathcal{A}$ and a challenger $\mathfrak{C}$. The adversary is considered capable of modifying public keys but does not have access to the master key of the Main Server (MS). The structure of the games demonstrating the scheme's security properties is described below:

1. **Confidentiality:** The PQLS-EVote is considered secure against adaptive chosen ciphertext attacks if no adversary $\mathcal{A}$ succeeds in the following game:
   Game-1: The challenger $\mathfrak{C}$ initiates the protocol by executing the $\text{Setup}(1^n)$ algorithm to generate the system parameters. The adversary $\mathcal{A}$ may then interact with the challenger by issuing the following types of queries:
   - *Public Key Query:* For a given identity I_i, $\mathcal{A}$ requests the corresponding public key, which is then returned by $\mathfrak{C}$.
   - *Partial Private Key Query:* $\mathcal{A}$ asks for a partial private key associated with identity I_i, and $\mathfrak{C}$ responds with the appropriate value.
   - *Private Key Extraction:* The adversary submits a query to obtain the complete private key associated with identity I_i, and $\mathfrak{C}$ responds by providing the requested key.
   - *Public Key Modification:* The $\mathcal{A}$ substitutes (R_i, Y_i), the original public key pair, with a newly chosen pair of values.
   - *Signcryption Query:* A request is made for signcrypting a message M, and $\mathfrak{C}$ returns the signcrypted ciphertext using the relevant algorithm.
   - *Unsigncryption Query:* A ciphertext is submitted by $\mathcal{A}$, and the challenger provides the corresponding output after executing the unsigncryption process.

   In the challenge phase, $\mathcal{A}$ submits two messages of equal length. The challenger then randomly chooses a bit $a \in \{0,1\}$ and generates the challenge ciphertext C based on message M_a. Notably, the adversary is restricted from querying the full private key corresponding to identity I_i whose public key remains unaltered during this phase.

   Following the challenge, $\mathcal{A}$ may continue issuing adaptive queries, but cannot request a partial private key for any identity whose public key was altered prior to

the challenge. Similarly, $\mathcal{A}$ cannot query the unsigncryption oracle using the challenge ciphertext C^*.

The adversary ultimately outputs a guess a^* for the value of a. If $a^* = a$, the adversary is deemed to have won the game. The advantage of the adversary in Game-1 is calculated as:

$$\mathrm{Adv}(\mathcal{A}) = \left|\Pr\left[a^* = a\right] - \frac{1}{2}\right|$$

2. **Unforgeability:** In this case, if Game-2 is not succeeded by any adversary, then the scheme offers resistance to adaptive chosen message attacks.

   Game-2: The game begins with running the $\mathrm{Setup}\left(1^n\right)$ algorithm by challenger $\mathfrak{C}$ to produce system parameters. During this phase, the adversary $\mathcal{A}$ may interact through various queries. Subsequently, $\mathcal{A}$ attempts to forge a valid ciphertext C^*.

   To ensure unforgeability, the adversary is not allowed to request the private key of the legitimate sender identity I_s nor is it permitted to query the unsigncryption oracle using C^*. If $\mathcal{A}$ produces a ciphertext that successfully decrypts to a valid message, the adversary wins the game. The advantage in Game-2 is defined as the probability that this event occurs.

14.4 PQLS-EVOTE: PROPOSED SCHEME

The proposed scheme is introduced in this section, which is strengthened using lattice-based cryptographic techniques. Initially, the lattice-based signcryption protocol is outlined in detail. Subsequently, the electronic voting process utilizing the proposed protocol is thoroughly discussed.

14.4.1 Lattice-Based Signcryption Protocol

There are four fundamental phases in our proposed protocol, which are detailed here:

14.4.1.1 Setup

- *EA* selects positive integers n, m, a large prime number q, and d such that $m \geq 5n \log q$, and a Gaussian parameter $\sigma \geq \beta$, where $\beta \geq O\left(\sqrt{n \log q}\right)$.
- And choose secure cryptographic hash functions are defined: $H_1 : \{0,1\}^* \rightarrow \mathbb{Z}_q^{m \times m}, H_2 : \{0,1\}^* \rightarrow \{0,1\}^d, H_3 : \{0,1\}^* \rightarrow \mathbb{Z}_q^n$.

14.4.1.2 Key Generation

- The Election Authority (EA) executes **TrapGen** (n, m, q) to generate matrix $A \in \mathbb{Z}_q^{n \times m}$ and a trapdoor basis B_A for the lattice $L^{\perp}(A)$.
- A voter V_i registers with identity I_i. The EA computes $D_i = H_1(I_i)$ and $R_i = AD_i$. Then, EA runs **BasisDel** (A, D_i, B_A, σ) to derive a basis $\mathcal{B}_i$ for $L^{\perp}(R_i)$.
- The EA applies the **SampPre** $(R_i, \mathcal{B}_i, \sigma, D_i)$ algorithm to derive M_i such that the relation $R_i M_i = D_i$ holds. Subsequently, the EA transmits R_i (serving as the partial public key) and M_i (serving as the corresponding partial private key) to the voter V_i.

- The voter selects matrices J_i and L_i with small norms and computes: $V_i = R_i L_i$, $Y_i = R_i J_i$, $X_i = M_i + J_i + L_i$. Here, Y_i and X_i are the voter's public and private keys, respectively.

14.4.1.3 Signcryption

- The voter V_s selects a message M (representing a vote) and a set of ring members $\mathrm{U} = \{I_1, I_2, \ldots, I_n\}$.
- Computes the aggregate public key $\mathbf{Y}' = \sum_{i=1}^{n} Y_i$ and computes $\boldsymbol{\mu} = H_2(\mathbf{k}, M)$, where $\mathbf{k}$ is the concatenation of columns of $\mathbf{Y}'$.
- For all $i \neq s$, sample $z_i \leftarrow D_\sigma^m$. For the actual voter s, compute $z_s = X_s \varsigma + g_s$ using rejection sampling, where $\varsigma = H_2\left(\sum R_i g_i \mu[i], M\right)$.
- The message M is concealed by computing $\gamma = M \oplus H_2(Y_r, \omega)$. Then, the following values are determined:

$$h_1 = R_r \varepsilon_1 \bmod q, \quad h_2 = (Y_r + D_r + V_r)\varepsilon_1 + \varepsilon_2 + \omega \left\lfloor \frac{q}{2} \right\rfloor \bmod q$$

where $\varepsilon_1, \varepsilon_2, \omega \leftarrow D_\sigma^m$ are sampled independently from the discrete Gaussian distribution.

- The voter prepares and transmits the signcrypted ballot $C = (\gamma, h_1, h_2, \mathbf{k}, U, \mathcal{S})$, where the signature component is given by $\mathcal{S} = (z_1, z_2, \ldots, z_r, \varsigma)$.

14.4.1.4 Unsigncryption

- The Tallying Authority computes $\omega' = h_2 - h_1 X_r$ and reconstructs the binary vector ω by thresholding each component of ω'.
- Decrypts the vote as $M = \gamma \oplus H_2(Y_r, \omega)$ and recomputes $\propto = H_2(\mathbf{k}, M)$.
- Validates the ciphertext by checking if: $\zeta = H_2\left(\sum_{i=1}^{n} R_i \mu[i] z_i - (D_s + Y_s + V_s)\right.$ $\left.\mu[s]\zeta, M\right) = \varsigma$, and $\|z_i\| < 2\sigma\sqrt{n}$ for all i.

14.4.2 Lattice-Based Signcryption Approach for Managing Electronic Voting

This section introduces three primary protocols involved in electronic voting: the registration protocol, the management protocol, and the access protocol. Each of these is designed using a secure signcryption mechanism grounded in lattice-based cryptography. Furthermore, the overall workflow of the electronic voting management system is illustrated in Figures 14.1 and 14.2 to enhance clarity and comprehension.

PQLS-EVote consists of a voter (V_i) with their voting data, the MS of the EA, the Election Commission (EC), and a group of election centers forming a ring-like structure. In the PQLS-EVote, first of all, the voter registers with the MS of the EA, which then provides partial keys using the voter's information. Then the voter generates their full public and private keys. The voter chooses a random value c_i, creates the signcrypted ciphertext, and sends it to the MS of the EA. After verifying the ciphertext, the MS makes this c_i public. Then the voter uses the (t, n) threshold scheme and uses the vote M as the

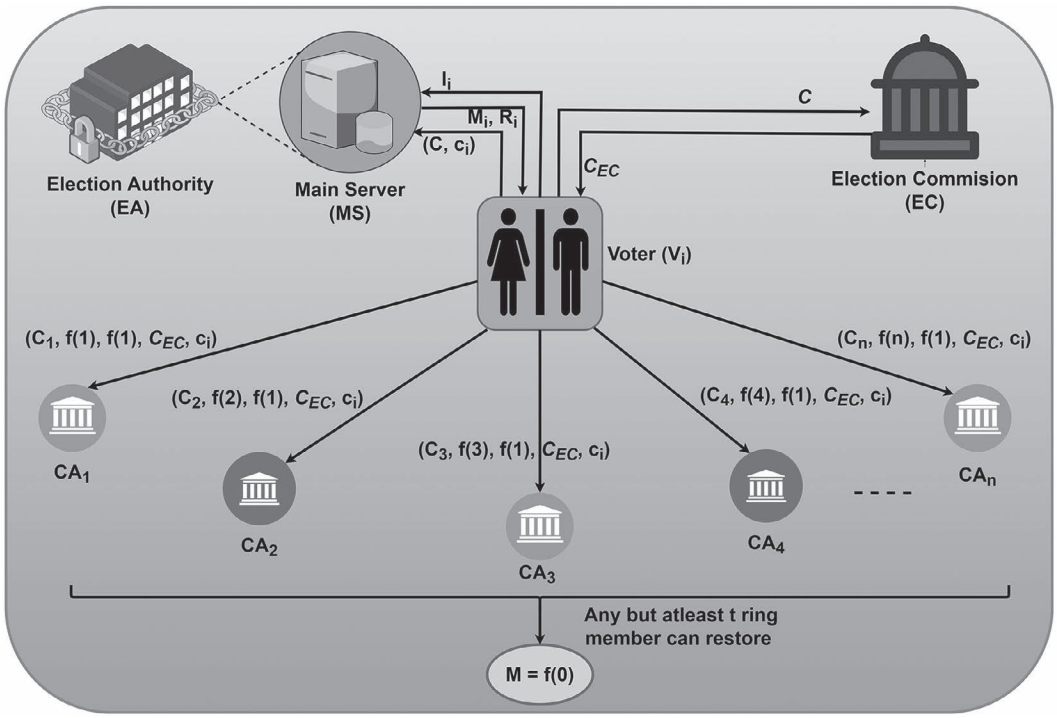


Figure 14.1 Proposed architecture of e-voting.

secret value of the scheme, selecting members for the ring structure. The voter uses the signcryption phase of the lattice-based signcryption protocol and creates a signcrypted ciphertext C for $f(1)$ and sends this C to the EC. The EC verifies it using the unsigncryption phase, stores $f(1)$ and c_i in the database, and sends back a ciphertext C_{EC} to the voter. After verifying the ciphertext, the voter calculates ciphertext C_{EC} of $f(k)$ and sends $(C_k, f(k), C_{EC}, c_i)$ to the election center EC_k. The details of each agreement are given one by one as follows.

i. **Registration Agreement:** Initially, the MS generates the required matrices $\mathbf{A}$ and B_A using the **TrapGen** (n, m, q) procedure. Following this, a voter V_i initiates the registration by providing their identity I_i to the MS. Upon successful verification, the MS proceeds to register the voter, as summarized in Table 14.2
ii. Based on the voter's information, the MS then derives the corresponding partial public and private keys as follows:

   Step 1: The MS first computes $D_i = H_1(I_i)$ and then evaluates $R_i = AD_i$. It executes the **BasisDel** (A, D_i, B_A, σ) procedure to derive a basis $\mathcal{B}$ for the lattice $L^{\perp}(R_i)$, where $\mathcal{B} \in \mathbb{Z}_q^{m \times m}$. Subsequently, MS invokes **SampleMatPre** $(R_i, \mathcal{B}, \sigma, D_i)$ to obtain a matrix M_i such that $R_i M_i = D_i$, with $M_i \in \mathbb{Z}_q^{m \times b}$ and a bounded norm $\|M_i\| \leq \sigma\sqrt{m}$. Finally, the R_i and the M_i are transmitted to the voter.

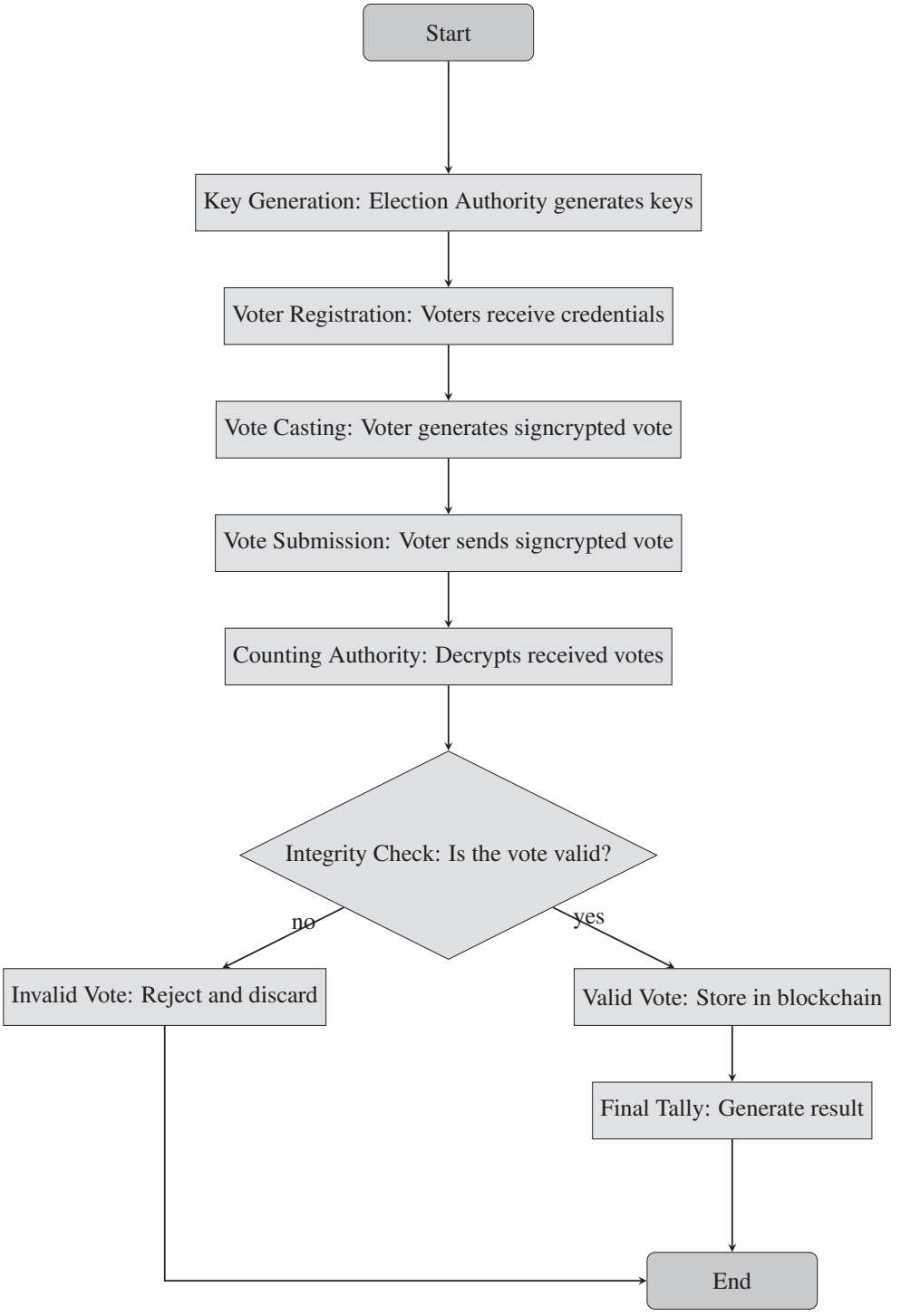


Figure 14.2 Flowchart for the PQLS-EVote.

Table 14.2 Description of the Registration Agreement Carried out between the Main Server (MS) and Voter (V_i) in the Context of the E-voting System

(MS)	(V_i)
Generate $(\mathbf{A}, B_A) \leftarrow \mathbf{TrapGen}(n, m, q)$.	$\xleftarrow[(Via\ secure-channel)]{Identity=\{I_i\}}$
Derive $D_i = H_1(I_i)$ and execute the $\mathbf{BasisDel}(A, D_i, B_A, \sigma)$ algorithm to obtain the trapdoor basis $\mathcal{B}$ of $L^{\perp}(R)$, where $\mathcal{B} \in \mathbb{Z}_q^{m \times m}$. Compute the partial public key, $R_i = AD_i$, of voter i. Then run $\mathbf{SampleMatPre}(R_i, \mathcal{B}, \sigma, D_i)$ to obtain $M_i \in \mathbb{Z}_q^{m \times b}$ such that $R_i M_i = D_i$ and $\lvert M_i \rvert \leq \sigma\sqrt{m}$. $\xrightarrow[(Via\ secure-channel)]{Partial\ key\ pair=\{R_i, M_i\}}$	
	Chooses random matrices $L_i \leftarrow \{-l, \ldots, 0, \ldots, l\}^{m \times b}$ and $J_i \leftarrow \{-j, \ldots, 0, \ldots, j\}^{m \times b}$ with $l, j \in \mathbb{Z}^+$, such that $\lVert L_i \rVert < l\sqrt{m}$ and $\lVert J_i \rVert < j\sqrt{m}$. Computes verification parameter $V_i = R_i L_i$, private key $X_i = M_i + J_i + L_i$, satisfying $\lVert X_i \rVert < (\sigma + b + c)\sqrt{m}$ and public key $Y_i = R_i J_i$. Picks a random bit string $\mathbf{w}_i \in \{0,1\}^*$ and evaluates $c_i = H_4(\mathbf{w}_i)$. Using the signcryption procedure, constructs the ciphertext C. $\xleftarrow[(Via\ secure-channel)]{(C,\ c_i)}$
Performs ciphertext verification via the unsigncryption mechanism and publishes c_i.	

Step 2: Once the voter rece ives the keys from the MS, they choose two random matrices $L_i \leftarrow \{-l, \ldots, 0, \ldots, l\}^{m \times b}$ and $J_i \leftarrow \{-j, \ldots, 0, \ldots, j\}^{m \times b}$, where $l, j \in \mathbb{Z}^+$ and $\lVert L_i \rVert < l\sqrt{m}, \lVert J_i \rVert < j\sqrt{m}$. Then, the voter computes $V_i = R_i L_i$ and derives their public key as $Y_i = R_i J_i \in \mathbb{Z}^{n \times b}$ and private key as $X_i = (M_i + J_i + L_i) \in \mathbb{Z}^{m \times b}$, ensuring the norm $\lVert X_i \rVert < (\sigma + b + c)\sqrt{m}$.

Step 3: The voter V_i chooses a binary string $\mathbf{w}_i \in \{0,1\}^*$ and computes $c_i = H_4(\mathbf{w}_i)$. This value c_i is then utilized during the signcryption step to generate a ciphertext C, and the pair (C, c_i) is sent to the MS. After receiving C, the MS performs unsigncryption to validate it and subsequently publishes c_i.

ii. Voting Authorization Agreement: Each voter V_i utilizes a confidential voting credential M as the secret in a (t, n) threshold secret-sharing scheme (Shamir 1979), generating evaluations $f(1), f(2), \ldots, f(n)$. The voter randomly selects several certified Counting Authorities (CAs). It is assumed that n such authorities collectively constitute the ring-shaped

configuration denoted by $U = \{\mathbf{I}_1,\ldots,\mathbf{I}_n\}$, as summarized in Table 14.3. The voter performs the following operations:

Step 1: Compute the aggregated vector $\mathbf{Y}' = \sum_{i=1}^{n} \mathbf{Y}_i = \left|\mathbf{y}'_1,\ldots,\mathbf{y}'_b\right|$, and set $\mathbf{k} = \left(\mathbf{y}_1'^{\top},\ldots,\mathbf{y}_b'^{\top}\right)$. Then derive the vector $\boldsymbol{\mu} = H_2\left(\mathbf{k}, f(1)\right)$, where $\boldsymbol{\mu} = (\propto_1, \propto_2,\ldots,\propto_q)$. The voter picks a random vector $g_i \in D_\sigma^m$ and computes the hash value $\varsigma = H_3\left(\sum_{i=1}^{n} R_i g_i \mu_i, f(1)\right)$.

Table 14.3 Summary of Voting Authorization Agreement between Voter (V_i) and Election Commission (EC)

(V_i)	(EC)
The voter generates shares $f(1), f(2),\ldots,f(n)$ from the confidential credential M using a threshold sharing scheme. Then calculates: $\mathbf{Y}' = \sum_{i=1}^{n} \mathbf{Y}_i = \left\|\mathbf{y}'_1,\ldots,\mathbf{y}'_b\right\|, \mathbf{k} = \left(\mathbf{y}_1'^{\top},\ldots,\mathbf{y}_b'^{\top}\right)$, and $\boldsymbol{\mu} = H_2\left(\mathbf{k}, f(1)\right)$. After selecting $g_i \in D_\sigma^m$, compute $\varsigma = H_3\left(\sum_{i=1}^{n} R_i g_i \mu_i, f(1)\right)$. Then compute $\gamma = f(1) \oplus H_3(Y_r, \omega)$ with $\omega \leftarrow D_\sigma^m$. Final steps include $h_1 = R_r \varepsilon_1 \bmod q$ and $h_2 = \left(\left(Y_r + D_r + V_r\right)\varepsilon_1 + \varepsilon_2 + \omega \left\lfloor \frac{q}{2} \right\rfloor\right) \bmod q$, using fresh samples $\varepsilon_1, \varepsilon_2 \leftarrow D_\sigma^m$. The final vote package is $C = (\gamma, h_1, h_2, \mathbf{k}, U, \mathbf{w}_i, \mathcal{S})$, where $\mathcal{S} = (z_1, z_2,\ldots, z_r, \zeta)$. $\xrightarrow[(Via\ secure-channel)]{C=\{(\gamma,h_1,h_2,\mathbf{k},U,\mathbf{w}_i,\mathcal{S})\}}$	
	The EC computes $\omega' = \left(\omega'_1, \omega'_2,\ldots,\omega'_\lambda\right) = h_2 - h_1 X_r$ for some $\lambda > k$. Then, for each $l = 1,\ldots,\lambda$, set $\omega_l = 0$ if $\omega'_l \in \left[-\frac{q}{4}, \frac{q}{4}\right]$; otherwise set $\omega_l = 1$. With $\omega = (\omega_1,\ldots,\omega_\lambda)$, compute $f(1) = \gamma \oplus H_3(Y_r, \omega)$. Then recalculate $\propto = H_2\left(\mathbf{k}, f(1)\right)$ and verify using $\zeta = H_3\left(\sum_{i=1}^{k} R_i \mu_i z_i - \left(D_s + Y_s + V_s\right)\mu_s \varsigma, f(1)\right)$. If $\zeta = \varsigma$ and $\|z_i\| < 2\sigma\sqrt{n}$, the ciphertext is considered valid. Then verify $c_i^{'} = H_4\left(\mathbf{w}_i\right)$ and check for matching public c_i. If not found, discard; otherwise, if c_i is new, store $\left(f(1), c_i\right)$. Then utilize X_{EC} to derive C_{EC} for future use. $\xleftarrow[(Via\ secure-channel)]{Ciphertext=\{C_{EC}\}}$
EC validates C_{EC} for integrity and accepts if correct.	

Step 2: Using rejection sampling as in Lyubashevsky (2012), the voter generates a tuple (z_i, ς) to decorrelate z_s and X_s. For all $i \neq s$, set $z_i = g_i$; for $i = s$, compute $z_s = X_s\varsigma + g_s$ with probability $min\left(\frac{D_\sigma^m}{ND_{X_s,\varphi}^m}, 1\right)$ and set $\mathcal{S} = (z_1, z_2, \ldots, z_r, \varsigma)$.

Step 3: Next, the voter computes $\gamma = f(1) \oplus H_3(Y_r, \omega)$, with $\omega \leftarrow D_\sigma^m$. Further, with $\varepsilon_1, \varepsilon_2 \leftarrow D_\sigma^m$, computations include $h_1 = R_r\varepsilon_1 \bmod q$ and $h_2 = \left((Y_r + D_r + V_r)\varepsilon_1 + \varepsilon_2 + \omega\left\lfloor\frac{q}{2}\right\rfloor\right) \bmod q$.

Step 4: Construct the final ciphertext $C = (\gamma, h_1, h_2, \mathbf{k}, U, \mathbf{w}_i, \mathcal{S})$ as signcrypted vote and transmit it securely to the EC.

Once the EC obtains the signcrypted ciphertext from the voter, the EC initiates a verification process to determine the authenticity of the received data through the following steps:

Step 1: The EC first computes the temporary vector $\omega' = \left(\omega_1', \omega_2', \ldots, \omega_\lambda'\right)$ using the formula:

$$\omega' = h_2 - h_1 X_r,$$

where $\lambda > n$. For each $k = 1, 2, \ldots, \lambda$, the value of ω_k is set as follows:

$$\omega_k = \begin{cases} 0 & \text{if } \omega_k' \in \left[-\frac{q}{4}, \frac{q}{4}\right], \\ 1 & \text{otherwise} \end{cases}$$

Step 2: Using the resulting vector $\omega = (\omega_1, \omega_2, \ldots, \omega_\lambda)$, compute:

$$f(1) = \gamma \oplus H_3(Y_r, \omega).$$

With this value of $f(1)$, calculate the hash:

$$\mu = H_2(\mathbf{k}, f(1)) = (\mu_1, \mu_2, \ldots, \mu_d).$$

Finally, evaluate:

$$\zeta = H_3\left(\sum_{i=1}^{n} R_i\mu_i z_i - (D_s + Y_s + V_s)\mu_s\varsigma, f(1)\right).$$

Step 3: If $\zeta = \varsigma$ and $\|z_i\| < 2\sigma\sqrt{n}$, then the received ciphertext C is declared valid and the vote M is accepted; otherwise, the ciphertext is considered invalid and discarded.

The EC discards the incoming data if the received signature is found to be invalid. Otherwise, it proceeds to compute: $c_i' = H_4(\mathbf{w}_i)$. Check whether a matching public value c_i exists such that $c_i = c_i'$. If not, the data is rejected. If they match, further check whether c_i is already recorded. If so, this indicates that voter V_i has already submitted a vote, and it is rejected to prevent duplication. If not found, insert c_i and $f(1)$ into the database. Then, using its secret key X_{EC}, the EC computes C_{EC} during the signcryption process and sends C_{EC} to the voter V_i.

iii. Voting Agreement: Upon receiving the signcrypted data C_{EC} from the EC, the voter proceeds as follows:

- If V_i verifies C_{EC} as authentic, they proceed to generate C_j targeting $f(j)$ by employing the identical signcryption technique used earlier for $f(1)$. The voter then sends $\left(C_j, f(1), f(j), C_{EC}, c_i\right)$ to the counting authority $CA_j\,(j \in U)$ via an anonymous communication channel. The CA_j verifies the signature and, if valid, makes public the tuple $\left(j, c_i, f(1)\right)$.
- If V_i notices that $\left(j, c_i, f(1)\right)$ has not been published, they raise a complaint by requesting $\left(f(1), C_{EC}\right)$. The MS then requests CA_j to publish the missing tuple.
- When the voter requests vote reconstruction, any t out of n CAs collaborate to compute the message M using the method described in Shamir (1979).

14.5 CORRECTNESS

The proposed scheme PQLS-EVote's correctness analysis is presented in this section, which is described in the following three parts.

1. By applying the discrete Gaussian distribution alongside the rejection sampling method, one can derive $z_s = X_s\varsigma + g_s$ with a probability at least $\min\left(\frac{D_\sigma^m}{ND_{X_s,\chi}^m}, 1\right)$. The resulting distribution of z_s closely follows the discrete Gaussian D_σ^m, exhibiting a statistical distance bounded by $2^{-\omega \log m}/N$. Furthermore, as shown in Lemma 3.3 of Lyubashevsky (2012), the vector z_s satisfies the norm bound $\|z_s\| < 2\sigma\sqrt{n}$ with overwhelming probability, namely, at least $1 - 2^{-n}$.
2. The authenticity of the signature is confirmed by checking the following relation.

$$\sum_{i=1}^{n} R_i z_i \mu_i - \left(D_s + Y_s + V_s\right)\mu_s\varsigma = \sum_{i=1}^{n} R_i g_i \mu_i.$$

This equation outlines the verification process, ensuring that the computed values satisfy the correctness condition.

$$\begin{aligned}\left(\sum_{i=1}^{n} R_i z_i \mu_i - \left(D_s + Y_s + V_s\right)\mu_s\varsigma\right) &= R_s\left(X_s\varsigma + g_s\right)\mu_s - \left(D_s + Y_s + V_s\right)\mu_s\varsigma + \sum_{i\in U-s} R_i z_i \mu_i \\ &= R_s\left(M_s + J_s + L_s\right)\mu_s\varsigma - \left(D_s + Y_s + V_s\right)\mu_s\varsigma + R_s g_s \mu_s + \sum_{i\in U-s} R_i g_i \mu_i \\ &= \sum_{i=1}^{n} R_i g_i \mu_i\end{aligned}$$

Hence,

$$\zeta = H_3\left(\sum_{i=1}^{n} R_i z_i \mu_i - \left(D_s + Y_s + V_s\right)\mu_s\varsigma\right), M\right) = H_3\left(\sum_{i=1}^{n} R_i g_i \mu_i g, M\right) = \varsigma$$

3. Verification of the unsigncryption procedure:

$$\begin{aligned}\omega' &= h_2 - h_1 X_r \\ &= (Y_r + D_r + V_r)\varepsilon_1 + \varepsilon_2 + \omega\left\lfloor\frac{q}{2}\right\rfloor - R_r\varepsilon_1 X_r \\ &= \varepsilon_2 + \omega\left\lfloor\frac{q}{2}\right\rfloor.\end{aligned}$$

where $\varepsilon_2 \leftarrow D_\sigma^n$ and $|\varepsilon_2| < \left\lfloor\frac{q}{4}\right\rfloor$. For $l = 1,2,\ldots,\lambda$, if $\omega_l^{'} \in \left[-\frac{q}{4},\frac{q}{4}\right]$ then set $\omega_l = 0$; otherwise, sets $\omega_l = 1$. The message can be extracted by applying $M \leftarrow \gamma \oplus H_2(Y_r,\omega)$ at the receiver's end.

14.6 SECURITY ANALYSIS

In this section, we provide a detailed security analysis of the proposed electronic voting scheme through formal theorems. The explanation is as follows.

Theorem 14.6.1. *The anonymity against the full-key exposure is satisfied by the PQLS-EVote.*

Proof: This proof illustrates how our PQLS-EVote preserves anonymity despite the exposure of the full key. There is a game between adversary $\mathcal{A}$ and challenger $\mathcal{C}$, which involves some steps like setup, signcryption query, partial private key query, private key query, challenge, and final guess by $\mathcal{A}$. Each step is explained as follows:

Setup: In this step, let the parameters n, m, and q are given. Challenger $\mathcal{C}$ computes $(A, B_A) \leftarrow \text{Setup}(1^n)$. Then $\mathcal{C}$ sends the ring public parameters $(rpp) = \{n, A\}$ to $\mathcal{A}$.

Signcryption Query: For given (I_i, M, U), the $\mathfrak{C}$ computes a signcrypted ciphertext C_{I_i} using signcryption phase and sends C_{I_i} to $\mathcal{A}$.

Partial Private Key Query: For given I_i, the challenger $\mathfrak{C}$ computes the partial private key M_i and sends it to the adversary $\mathcal{A}$.

Private Key Query: For given M_i, the challenger $\mathfrak{C}$ computes the private key X_i and sends it to the adversary $\mathcal{A}$.

Challenge: For a message M, a ring set U, and the public keys of two signers I_0, I_1, the challenger $\mathfrak{C}$ arbitrarily selects $b \in \{0,1\}$, computes $C_{I_b} \leftarrow \text{Ring-Sign}(\text{params}, I_b, M, U)$, and sends C_{I_b} to adversary $\mathcal{A}$.

Guess: Adversary $\mathcal{A}$ outputs b'. In the previous process, the signcrypted ciphertext of the two signers I_0 and I_1 are C_{I_0} and C_{I_1}, respectively. Since C_{I_0} and C_{I_1} are computed using a random matrix $J_i \in \mathbb{Z}_q^{m \times b}$ (since $Y_i = R_i J_i$) and a random vector $g_i \in \mathbb{D}_\sigma^m$, C_{I_0} and C_{I_1} are indistinguishable. Therefore, the advantage of adversary $\mathcal{A}$ to win the game is negligible. This scheme satisfies anonymity against full-key exposure.

Theorem 14.6.2. *(Confidentiality) The PQLS-EVote guarantees confidentiality, assuming that the adversary $\mathcal{A}$ cannot breach the confidentiality of the ring signcryption based on lattice-based assumptions in the ROM. If an adversary manages to succeed in Game-1 with*

advantage α*, consequently, the challenger* $\mathfrak{C}$ *is capable of resolving the Decision Ring LWE (D-RLWE) problem with an advantage of at least* $\alpha' \geq \alpha / eq_k$*, where* $q_k = (q_1 + q_2 + q_3)$ *represents the total number of queries for private key, public key replacement, and partial private key, and* e *is the base of the natural logarithm.*

Proof: To prove confidentiality, we construct a simulator $\mathfrak{C}$ that embeds an instance of the D-RLWE problem and interacts with $\mathcal{A}$ as a subroutine. The challenger maintains initially empty sets: list1, list2, and list3, for tracking hash and key queries. $\mathfrak{C}$ selects a prime q and an integer m such that $m \geq 5n \log q$. It executes the $\text{Setup}(1^n)$ algorithm to generate a matrix $A \in \mathbb{Z}_q^{n \times m}$ and base matrix $B_A \in \mathbb{Z}_q^{m \times m}$. Then, $\mathfrak{C}$ provides the tuple $\theta = \{\sigma, \beta, A, H_1, H_2, H_3, H_4\}$ to $\mathcal{A}$.

Adversary $\mathcal{A}$ may issue the following adaptive queries:

H_1 **queries:** For an identity I_i, $\mathfrak{C}$ checks list1. If found, it returns D_i; otherwise, it samples random $F_i \in \mathbb{Z}_q^{n \times k}$, returns it, and stores (I_i, F_i) in list1.

H_2 **queries:** For a queried tuple, if present in list2, $\mathfrak{C}$ returns the corresponding $\propto$. Otherwise, a random $\propto \in \mathbb{Z}_2^d$ is returned and added to list2.

H_3 **queries:** $\mathcal{A}$ requests H_3 on some inputs. If a matching tuple exists in list3, the stored ς is returned. Else, a random $\varsigma \in \{0,1\}^l$ is chosen and the tuple $(R_i, g_i, \mu_i, f(1), \varsigma)$ is saved in list3.

H_4 **queries:** If the queried input is found in list4, $\mathfrak{C}$ returns the corresponding c_i. Otherwise, a random $c_i \in \mathbb{Z}_q^n$ is selected, returned, and stored in list4.

Public key query: On receiving a request for the public key of identity I_i, $\mathfrak{C}$ checks list5. If absent, it samples $D_i \in \mathbb{Z}_q^{n \times n}, L_i \leftarrow \{-l, \ldots, 0, \ldots, l\}^{m \times b}$, and $J_i \leftarrow \{-j, \ldots, 0, \ldots, j\}^{m \times b}$, then computes $R_i = AD_i, Y_i = R_i J_i, V_i = R_i L_i$ and returns (R_i, Y_i) while storing (R_i, Y_i, J_i, V_i) in list5.

Partial private key query: For a partial private key of identity I_i, if $I_i = I^*$ (challenge identity), $\mathfrak{C}$ aborts; otherwise, it chooses random $M_i \in \mathbb{Z}_q^{m \times b}$ and stores it in list4.

Private key query: For I_i, if $I_i = I^*$, $\mathfrak{C}$ aborts. Otherwise, it returns (J_i, L_i) from list5.

Public key replacement: If $\mathcal{A}$ tries to replace the public key of identity $I_i = I^*$, $\mathfrak{C}$ aborts. Otherwise, it replaces the entry in list5 with (R_i^*, Y_i^*).

Signcryption queries: For message $f(1)$:

1. Compute $\mathbf{Y}' = \sum_{i=1}^{n} \mathbf{Y}_i = \left[\mathbf{y}_1', \ldots, \mathbf{y}_b'\right]$, then $\mathbf{k} = \left(\mathbf{y}_1'^{\top}, \ldots, \mathbf{y}_b'^{\top}\right)$ and $\propto = H_2(\mathbf{k}, f(1))$.
2. Choose $g_i \in D_\sigma^m$ randomly and compute $\varsigma = H_3\left(\sum_{i \in [n]} R_i g_i \mu_i, f(1)\right)$.
3. Using rejection sampling (Lyubashevsky 2012), define:

$$z_i = \begin{cases} g_i, & i \neq s \\ X_s \varsigma + g_s, & i = s \text{ with prob. } \min\left(\dfrac{D_\sigma^m}{N D_{X_s, \varphi}^m}, 1\right) \end{cases}$$

where N is a constant and set $\mathcal{S} = (z_1, z_2, \ldots, z_r, \varsigma)$.

4. Generate $\omega \leftarrow D_\sigma^m$, compute $\gamma = f(1) \oplus H_3(Y_r, \omega)$, where $\varepsilon_1, \varepsilon_2 \leftarrow D_\sigma^m$ and then:

$$h_1 = R_r \varepsilon_1 \bmod q, \quad h_2 = (Y_r + D_r + V_r)\varepsilon_1 + \varepsilon_2 + \omega \lfloor q/2 \rfloor \bmod q$$

5. The final form of the ciphertext is $C = (\gamma, h_1, h_2, \mathbf{k}, U, \mathbf{w}_i, \mathcal{S})$.

Unsigncryption Queries: Given a ciphertext C, $\mathcal{A}$ initiates an unsigncryption request, to which $\mathfrak{C}$ responds as follows:

1. The EC computes $\omega' = (\omega_1', \omega_2', \ldots, \omega_\lambda') = h_2 - h_1 X_r$, where $\lambda > n$. For each $k = 1, 2, \ldots, \lambda$, if $\omega_k' \in \left[-\frac{q}{4}, \frac{q}{4}\right]$, the receiver sets $\omega_k = 0$; otherwise, $\omega_k = 1$.
2. Using the obtained $\omega = (\omega_1, \omega_2, \ldots, \omega_\lambda)$, the value $f(1)$ is derived as $f(1) = \gamma \oplus H_3(Y_r, \omega)$. With $f(1)$, compute the hash $H_2(\mathbf{k}, f(1)) = \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$. Then evaluate:

$$\zeta = H_3\left(\sum_{i=1}^{n} R_i \mu_i z_i - (D_s + Y_s + V_s)\mu_s \varsigma, f(1)\right).$$

3. If both conditions $\zeta = \varsigma$ and $\|z_i\| < 2\sigma\sqrt{n}$ hold, then the ciphertext is accepted and the voting information M is retrieved. Otherwise, it is discarded as invalid.

During the challenge phase, $\mathcal{A}$ outputs $\left(\{f_0(1), f_1(1)\}, I_s^\varphi, I_r^\varphi\right)$, where I_s^φ and I_r^φ are the identifiers for the signcrypter and the recipient, respectively. The adversary is not permitted to request the private key of I_r^φ before this phase. If $I_r^\varphi \neq I^\varphi$, then $\mathfrak{C}$ aborts the game. Otherwise, it proceeds by randomly selecting a bit $a \in \{0,1\}$ and generating a challenge ciphertext C^* corresponding to message $f_a(1)$:

1. Define $\mathbf{Y}' = \sum_{i=1}^{n} \mathbf{Y}_i = [\mathbf{y}_1', \ldots, \mathbf{y}_b']$, and form $\mathbf{k} = (\mathbf{y}_1'^\top, \ldots, \mathbf{y}_b'^\top)$. Compute the hash $\boldsymbol{\mu} = H_2(\mathbf{k}, f_a(1)) = (\alpha_1, \alpha_2, \ldots, \alpha_d)$. Select $g_i^* \in D_\sigma^m$ randomly and compute $\varsigma^* = H_3\left(\sum_{i=1}^{n} R_i^* g_i^* \mu_i, f_a(1)\right)$.
2. Using rejection sampling (Lyubashevsky 2012), $\mathfrak{C}$ obtains (z_i^*, ς^*) to reduce the correlation between z_s^* and X_s^*. If $i \neq s$, set $z_i^* = g_i^*$. Else, compute $z_s^* = X_s^* \varsigma^* + g_s^*$ with $\min\left(\frac{D_\sigma^m}{N D_{X_s^*, \varphi}^m}, 1\right)$ probability. If a valid output is not produced, repeat this step until one is.
3. Compute $\gamma^* = f_a(1) \oplus H_3(Y_r^*, \omega^*)$, with $\omega^* \leftarrow D_\sigma^m$. Also compute:

$$h_1^* = R_r^* \varepsilon_1^* \bmod q, \quad h_2^* = (Y_r^* + D_r^* + V_r^*)\varepsilon_1^* + \varepsilon_2^* + \omega^* \left\lfloor \frac{q}{2} \right\rfloor \bmod q.$$

4. Assemble the challenge ciphertext as $C^* = (\gamma^*, h_1^*, h_2^*, \mathbf{k}, U, \mathcal{S}^*)$.

Subsequently, $\mathcal{A}$ proceeds with adaptive queries as in the earlier stage but is prohibited from accessing the private key of I_r or requesting the unsigncryption of C^*. Ultimately, $\mathcal{A}$ outputs a guess a^* for a.

If $a^* = a$, and since $\mathfrak{C}$ knows that $\left(h_1^*, R_i^*\right)$ and $\left(h_2^*, Y_i^*\right)$ are sampled from the distribution $\mathbf{A}_{s,\chi}$, then by choosing $\varepsilon_1^*, \varepsilon_2^* \leftarrow D_\sigma^m$, $\mathfrak{C}$ solves an instance of the RLWE problem.

Probability Analysis: Let α' denote the adversary's advantage in distinguishing RLWE challenge instances, defined by the probability of three concurrent events:

- A_1: The simulation does not abort.
- A_2: The chosen identity I^* matches the receiver's identity I_r^*.
- A_3: The adversary successfully distinguishes the challenge ciphertext.

The probability of success for $\mathcal{A}$ conditioned on the simulation not aborting is given by:

$\Pr\left[A_3 \mid A_2 \cap A_1\right] \geq \alpha$ (Yu et al. 2022).

For $I^* = I_r$, assuming μ is the probability, we derive:

$$Pr\left[A_1\right] = \mu^{q_1+q_2+q_3}, \quad Pr\left[A_2 \mid A_1\right] = (1-\mu),$$

with $\mu = 1 - \dfrac{1}{1-(q_1+q_2+q_3)}$.

Thus, the overall distinguishing advantage becomes:

$$\begin{aligned}\alpha' &= Pr\left[A_1 \cap A_2 \cap A_3\right] \\ &= Pr\left[A_1\right] \cdot Pr\left[A_2 \mid A_1\right] \cdot Pr\left[A_3 \mid A_2 \cap A_1\right] \\ &\geq \mu^{q_1+q_2+q_3} \cdot (1-\mu) \cdot \alpha \\ &\geq \frac{\alpha}{e \cdot q_k},\end{aligned}$$

where $q_k = q_1 + q_2 + q_3$, and e is the base of the natural logarithm. The values q_1, q_2, and q_3 correspond to the number of partial private key queries, full private key requests, and public key replacement queries, respectively. This completes the proof that our electronic voting system maintains confidentiality based on the hardness of the D-RLWE assumption.

Theorem 14.6.3. *(Unforgeability) Let us assume there exists an adversary $\mathcal{A}$ capable of compromising the security of the PQLS-EVote scheme in the random oracle setting with a non-negligible advantage α. Then, a challenger $\mathfrak{C}$ can solve an instance of the SIS problem with an advantage of at least α', where $\alpha' \geq \alpha\left(1 - 2^{-\omega \log m}\right)$.*

Proof: We begin by assuming that adversary $\mathcal{A}$ is provided with a hard instance of the SIS problem, where the objective is to find a non-zero vector $x \in \mathbb{Z}_q^n$ such that $Ax \equiv 0 \bmod q$ and $\|x\| < \beta$. Initially, $\mathcal{A}$ makes a query to the random oracle H_1 for a voter identity V_i. Challenger $\mathfrak{C}$ sets up and returns $\left(A, B_A, H_1, H_2, H_3, H_4\right)$ along with a configuration $\theta = \left\{\sigma, \beta, A, H_1, H_2, H_3, H_4\right\}$ to adversary $\mathcal{A}$. The adversary then proceeds with a sequence of adaptive queries similar to those in the first phase of Theorem 14.6.2.

Eventually, $\mathcal{A}$ forges a ciphertext $C^* = \left(\gamma^*, h_1^*, h_2^*, \mathbf{k}, U, \mathcal{S}^*\right)$. Note that $\mathcal{A}$ must not request the private key corresponding to the identity V_S^* during this phase. If the identity V_i^* queried is not equal to V^*, $\mathfrak{C}$ aborts the game; otherwise, $\mathfrak{C}$ creates another ciphertext $C' = \left(\gamma', h_1', h_2', \mathbf{k}, U, \mathcal{S}'\right)$.

Applying the forking lemma (Lyubashevsky 2012), we obtain:

$$R_j \mu_j z_j^* - \left(D_s^* + Y_s^* + V_s^*\right)\mu_s \varsigma^* = R_j \mu_j z_j' - \left(D_s^* + Y_s^* + V_s^*\right)\mu_s \varsigma'$$

where

- EC_s^* is the partial private key for voter identity V_s^*,
- J_s^* and L_s^* are secret shares for the same identity,
- $D_s^* = R_s^* EC_s^*, Y_s^* = R_s^* J_s^*, V_s^* = R_s^* L_s^*$,
- $\varsigma^* \neq \varsigma'$.

Rewriting the equation, we get:

$$R_j \mu_j \left(z_j^* - z_j'\right) + R_s^* EC_s^* \mu_s \left(\varsigma' - \varsigma^*\right) + R_s^* J_s^* \mu_s \left(\varsigma' - \varsigma^*\right) + R_s^* L_s^* \mu_s \left(\varsigma' - \varsigma^*\right) = 0 \tag{14.1}$$

Given the bounds:

$$\left\|z_j^*\right\|, \left\|z_j'\right\| \leq 2\sigma\sqrt{n}, \quad \left\|\varsigma^*\right\|, \left\|\varsigma'\right\| \leq f, \quad \left\|J_s^*\right\| < b\sqrt{n}, \quad \left\|L_s^*\right\| < c\sqrt{n}, \quad \left\|EC_s^*\right\| < \sigma\sqrt{n}$$

We get the inequality:

$$\left\|z_j^* - z_j'\right\| + \left\|EC_s^*\left(\zeta' - \zeta^*\right) + J_s^*\left(\zeta' - \zeta^*\right) + L_s^*\left(\zeta' - \zeta^*\right)\right\| \leq 4\sigma\sqrt{n} + 2f\sigma\sqrt{n} + 2fb\sqrt{n} + 2fc\sqrt{n}$$

Hence, $\mathfrak{C}$ produces:

$$\left\|z_j^* - z_j'\right\| + \left\|EC_s^*\left(\varsigma' - \varsigma^*\right) + J_s^*\left(\varsigma' - \varsigma^*\right) + L_s^*\left(\varsigma' - \varsigma^*\right)\right\| \neq 0$$

with a non-negligible probability.

Once $\mathcal{A}$ outputs Y_s^*, $\mathfrak{C}$ can compute EC_s^* (the partial private key) and deduce the related secrets. According to Lemma 4.5 of Lyubashevsky (2012), there exists another valid key $EC_s' \neq EC_s^*$ such that $R_s^* EC_s^* = R_s^* EC_s'$ with high probability $\mathcal{P} \geq 1 - 2^{-100}$.

Therefore, a new version of Equation (14.1) will yield:

$$R_j \mu_j \left(z_j^* - z_j'\right) + R_s^* EC_s' \mu_s \left(\varsigma' - \varsigma^*\right) + R_s^* J_s^* \mu_s \left(\varsigma' - \varsigma^*\right) + R_s^* L_s^* \mu_s \left(\varsigma' - \varsigma^*\right) \neq 0$$

As $\mathfrak{C}$ is unaware of the specific partial key employed by $\mathcal{A}$ to construct the forgery, it chooses one at random. Therefore:

$$Pr\left[R_j \mu_j \left(z_j^* - z_j'\right) + R_s^* EC_s^* \mu_s \left(\zeta' - \varsigma^*\right) + R_s^* J_s^* \mu_s \left(\varsigma' - \varsigma^*\right) + R_s^* L_s^* \mu_s \left(\varsigma' - \varsigma^*\right) \neq 0\right] \geq \frac{1}{2}$$

Probability Analysis: The probability α' that $\mathfrak{C}$ solves the SIS problem satisfies:

$$\alpha' \geq \alpha\left(1 - 2^{-\omega \log m}\right)$$

which is negligible as per Lyubashevsky (2012). Thus, the PQLS-EVote scheme remains secure under the defined model in the context of electronic voting systems.

Theorem 14.6.4. *(Unconditional anonymity) Suppose that under the ROM, an adversary $\mathcal{A}$ cannot compromise the unconditional anonymity of a ring signcryption mechanism with probability advantage α. Then, the proposed PQLS-EVote framework also guarantees unconditional anonymity.*

Proof: To establish this, the challenger $\mathfrak{C}$ first executes the initialization routine, obtaining system parameters $(A, B_A, H_1, H_2, H_3, H_4)$ and provides $\eta = (A, H_1, H_2, H_3, H_4)$ to the adversary $\mathcal{A}$. The adversary then issues a sequence of queries as outlined in the initial phase of Theorem 14.6.2. Ultimately, $\mathcal{A}$ produces $\left(C^*, I_{s1}^*, I_{s2}^*\right)$ which are passed to $\mathfrak{C}$.

The challenger then chooses a random bit $\kappa \leftarrow \{0,1\}$, random samples $g_i \leftarrow D_\sigma^n$, and a binary string $\varsigma^* \leftarrow \{0,1\}^*$. For each $i \neq s_\kappa$, it sets $z_i^* = g_i$; for $i = s_\kappa$, it computes $z_{s_\kappa}^* = X_{s_\kappa}\varsigma^* + g_{s_\kappa}$. Consequently, $\mathfrak{C}$ constructs the ciphertext $C^* = \left(h_1, h_2, \gamma, \mathbf{k}, U, \left\{z_i^*\right\}, \varsigma^*\right)$, which corresponds to the identity $I_{S_\kappa}^*$.

According to Theorem 3.4 in Lyubashevsky (2012), the statistical distance between the Gaussian distribution D_σ^n and $z_{s_\kappa}^*$ is bounded by $\frac{2^{-\omega \log m}}{N}$, indicating that $C^* = \left(h_1, h_2, \gamma, \mathbf{k}, U, \left\{z_i^*\right\}, \varsigma^*\right)$ is computationally indistinguishable from $\left(D_\sigma^n\right)_{l+1}$.

Analogously, for a separate ciphertext C', the challenger sets $z_i^{'} \leftarrow D_\sigma^n$ for $i \neq s_{1-\kappa}$ and $z_{s_{1-\kappa}}^{'} = X_{s_{1-\kappa}}\varsigma' + g_{s_{1-\kappa}}$ for $i = s_{1-\kappa}$, with $\varsigma' \leftarrow \{0,1\}^*$. This results in $C' = \left(h_1, h_2, \gamma, \mathbf{k}, U, \left\{z_i'\right\}, \varsigma'\right)$, also indistinguishable from the ideal Gaussian distribution.

Therefore, since C^* and C' cannot be distinguished, the ring signcryption process inherently provides unconditional anonymity, making it suitable for privacy-preserving.

The PQLS-EVote's ability to resistant against replay attacks can also be demonstrated by examining the following theorem:

Theorem 14.6.5. *The resistance against replay attacks is satisfied by the PQLS-EVote.*

Proof: Every signer in our scheme creates a signcrypted ciphertext using a combination of private key and public key. Additionally, it periodically refreshes the private/public key pairs, making them useless while changing the ring members. Therefore, even if the attacker delivers a legitimate message-signature pair, it will be declared invalid. The attacker's device won't receive the data as a result. Consequently, replay attacks cannot succeed against the PQLS-EVote.

14.7 PERFORMANCE ANALYSIS

The performance assessment of the proposed scheme, PQLS-EVote, is presented in this section by comparing it with existing related protocols. The evaluation focuses on both the achieved security properties and the associated computational overhead, particularly in terms of the complexity of time.

14.7.1 Security Features Evaluation

This subsection presents a comparative evaluation of the proposed scheme PQLS-EVote with existing approaches proposed by Yu et al. (2022), Guo and Deng (2020), Cai et al. (2020), and

Table 14.4 Comparison of Security Features across Various Schemes

Security Feature	*Ref.Yu et al. (2022)*	*Ref. Guo and Deng (2020)*	*Ref. Cai et al. (2020)*	*Ref.Yu and Wang (2021)*	*Ref.Yu and Wang (2023)*	*PQLS-EVote*
Unforgeability	✓	✓	✓	✓	✓	✓
Confidentiality	✓	✓	✓	✓	✓	✓
Full-key anonymity	✓	✓	×	✓	✓	✓
Threshold signcryption	×	×	×	✓	✓	✓
Safe from quantum attacks	✓	×	×	×	✓	✓
Decentralization	×	×	×	×	✓	✓
Efficiency	**Poor**	**Poor**	**Poor**	**Moderate**	**Poor**	**Good**

Note: ✓ indicates that the feature is supported; × indicates that it is not supported. Ref. stands for Reference.

Yu and Wang (2021, 2023). The assessment focuses on key security attributes vital to cryptographic systems, namely unforgeability, confidentiality, threshold signcryption, resistance to quantum adversaries, decentralization, and overall efficiency. The feature-wise comparison is summarized in Table 14.4, demonstrating that the PQLS-EVote offers a well-rounded and robust security foundation for electronic voting environments.

From Table 14.4, it is evident that the schemes in Guo and Deng (2020) and Cai et al. (2020) neither provided protection against quantum adversaries nor support decentralization. Although the protocol proposed by Yu et al. (2022) offered post-quantum security, it does not facilitate a decentralized structure. The scheme by Yu and Wang (2021) supported threshold signcryption but lacks both quantum resistance and decentralization. Similarly, the scheme proposed by Yu and Wang (2023) ensured quantum security and supports threshold signcryption, yet it does not offer a decentralized framework and is rated poorly in terms of efficiency. In contrast, the PQLS-EVote is the only one that simultaneously ensures all critical security features, including unforgeability, confidentiality, anonymity against full-key exposure, threshold signcryption, quantum resilience, decentralization, and operational efficiency. Therefore, it demonstrates a comprehensive and superior solution for secure electronic voting environments.

14.7.2 Computation Time Analysis

This section evaluates the computational efficiency of the proposed protocol by examining the types and number of cryptographic operations required during its execution. The overall processing time is directly influenced by these operations. Specifically, the following notations are used to represent the time complexities: T_e represents the time for exponentiation, T_{sm} denotes the time for scalar multiplication, T_h stands for the time required to perform a hash function, T_{bl} refers to the bilinear pairing computation time, T_{isa} indicates the time taken for image sampling algorithm, T_{gsa} is the time associated with Gaussian sampling algorithm, and T_{mv} denotes the time required for vector-matrix multiplication. As suggested in Yu et al. (2022), the running time for exponential operation is $6.82\ ms$, for scalar multiplication is $0.05\ ms$, for hash operation is $16.59\ ms$, for bilinear operation is $21.6\ ms$, for image sampling algorithm is $35.42\ ms$, for Gaussian sampling algorithm is $23.03\ ms$, and for vector-matrix multiplication is $5.32\ ms$. By leveraging the execution times of individual cryptographic functions, we estimated the overall time complexity of

Table 14.5 Execution of Time Comparison across Existing Schemes (in ms)

Schemes	*Signcryption time*	*Unsigncryption time*	*Total time*
Ref. Yu et al. (2022)	$(4r+3)T_{mv}+4T_h+T_{isa}=1181.74$	$(r+1)T_{mv}+(r+2)T_h=1134$	$TT=(5r+4)T_{mv}+(r+6)T_h+T_{isa}=\mathbf{2315.74}$
Ref. Guo and Deng (2020)	$(2r+3)T_h+T_{bl}+(2r+4)T_{sm}=1735.57$	$(r+2)T_h+3T_{bl}+(2r+1)T_{sm}=932.53$	$TT=(3r+5)T_{sm}+4T_{bl}+(4r+5)T_h=\mathbf{2668.10}$
Ref. Cai et al. (2020)	$3rT_{sm}+(2r+4)T_h+T_{bl}=1754.46$	$rT_{sm}+(r+3)T_h+3T_{bl}=946.57$	$TT=4rT_{sm}+(4r+7)T_h+4T_{bl}=\mathbf{2704.03}$
Ref. Yu and Wang (2021)	$4T_{bl}+(4r+1)T_e=1457.22$	$4T_{bl}+T_e=93.22$	$8T_{bl}+(4r+2)T_e=\mathbf{1550.22}$
Ref. Yu and Wang (2023)	$rT_{isa}+(2r+1)T_{mv}=2840.32$	$4T_{mv}=21.28$	$rT_{isa}+(2r+5)T_{mv}=\mathbf{2861.60}$
PQLS-EVote	$(2r+2)T_{mv}+4T_{gsa}+3T_h+T_{isa}=719.95$	$(2r+1)T_{mv}+T_{gsa}+3T_h+T_{isa}=645.54$	$TT=(4r+3)T_{mv}+5T_{gsa}+6T_h+2T_{isa}=\mathbf{1365.49}$

Note: r denotes the number of ring members, set to 50, and 'Ref.' stands for Reference.

our PQLS-EVote in comparison with existing approaches. The time complexity of various schemes (Yu et al. 2022; Guo and Deng 2020; Cai et al. 2020; Yu and Wang 2021, 2023) and the PQLS-EVote is illustrated in Table 14.5, where r represents the number of members in the ring. Table 14.5 presents a comparative analysis of computational durations across various signcryption schemes, highlighting the time consumed during signcryption, unsigncryption, and the total process execution. The results, presented in milliseconds (ms), showcase the superior efficiency of our proposed scheme PQLS-EVote in terms of total time compared to the referenced schemes (Yu et al. 2022; Guo and Deng 2020; Cai et al. 2020; Yu and Wang 2021, 2023).

Based on the computed values, comparative plots are generated illustrating the performance of the proposed protocol against existing schemes (Yu et al. 2022; Guo and Deng 2020; Cai et al. 2020; Yu and Wang 2021, 2023). Specifically, Figure 14.3 displays the computation time required during the signcryption phase, Figure 14.4 corresponds to the unsigncryption phase, and Figure 14.5 summarizes the overall computational cost. It is evident from (r) increases, both the signcryption and unsigncryption times grow proportionally. However, the rate of increase in the PQLS-EVote remains significantly more controlled compared to the others. Furthermore, Figure 14.5 highlights that our scheme consistently achieves the lowest total computational time across all evaluated protocols. Notably, the PQLS-EVote achieves a total execution time of 1365.49 *ms*, outperforming Yu et al. (2022) (2315.74 *ms*), Guo and Deng (2020) (2668.10 *ms*), Cai et al. (2020) (2704.03 *ms*), Yu and Wang (2021) (1550.22 *ms*), and Yu and Wang (2023) (2861.60 *ms*). These results demonstrate that the proposed approach achieves better computational efficiency, especially in environments where performance is critical, such as electronic voting systems.

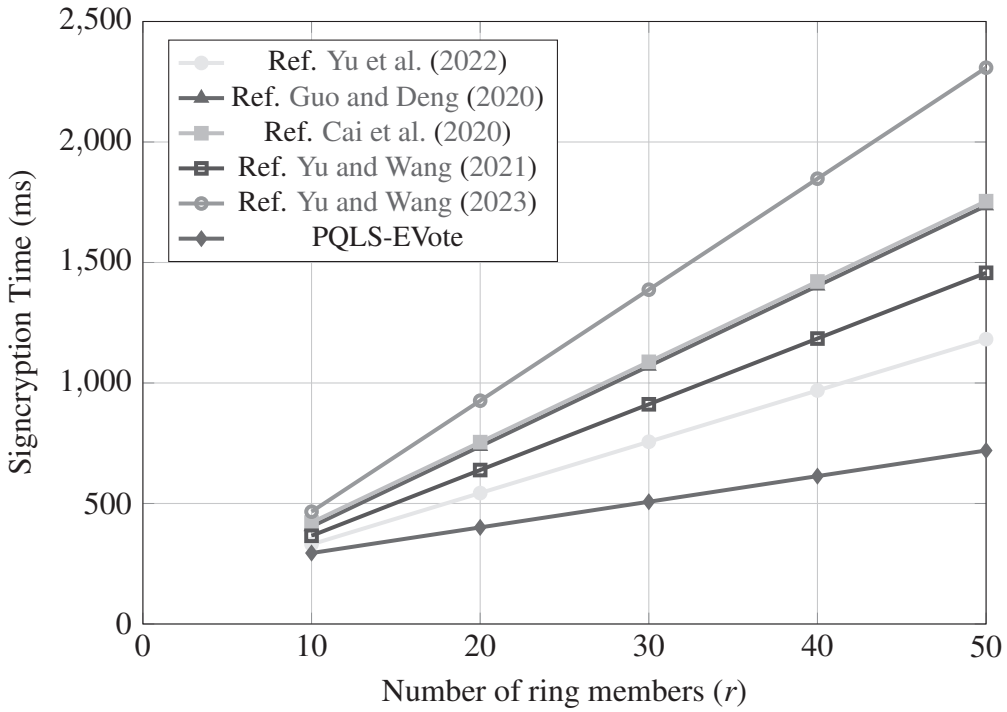


Figure 14.3 Comparison of signcryption time.

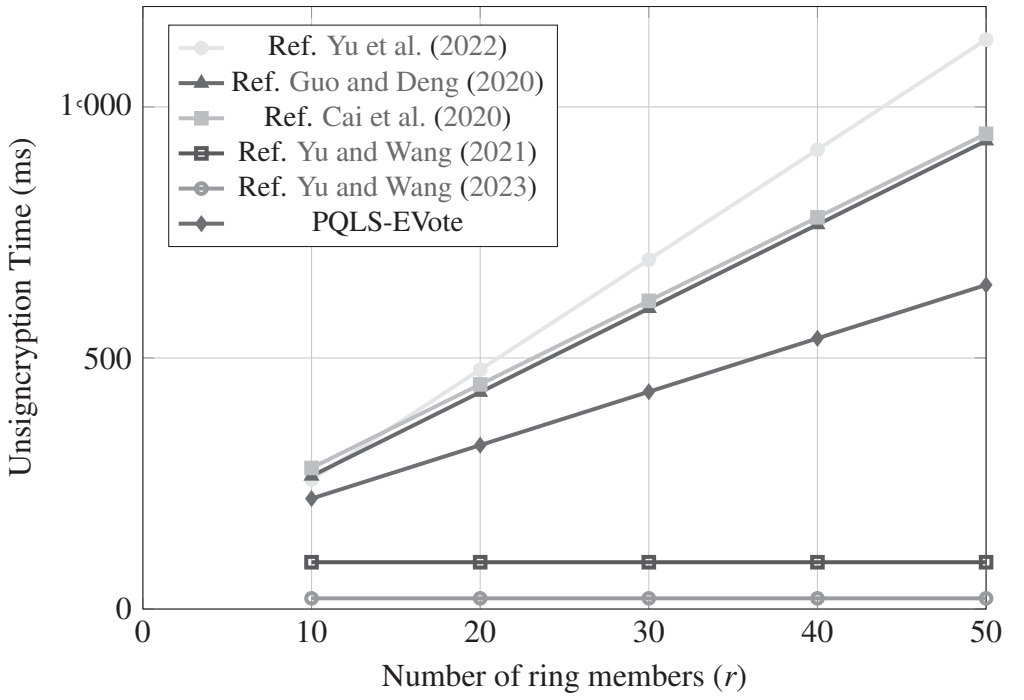


Figure 14.4 Comparison of unsigncryption time.

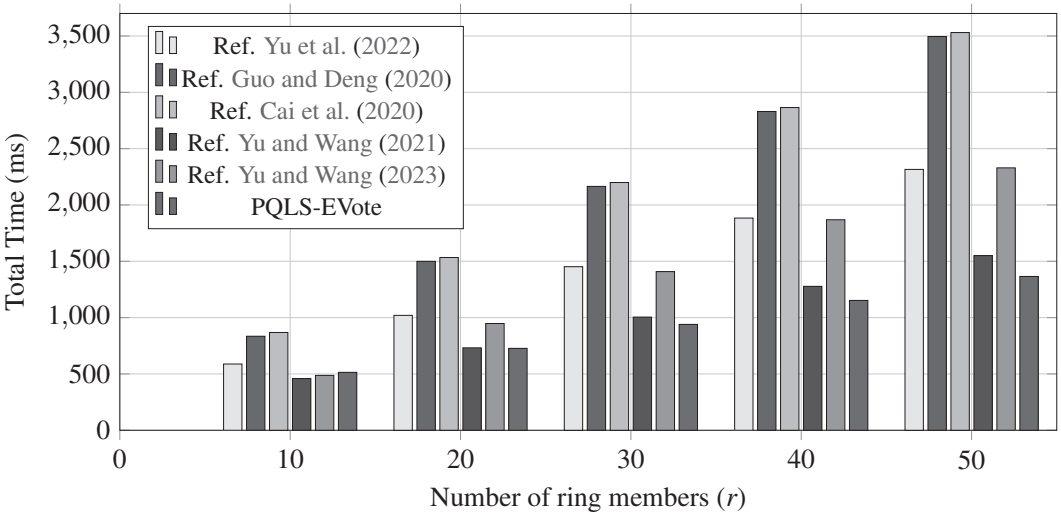


Figure 14.5 Comparison of total time.

14.8 RESULTS AND DISCUSSION

This section presents the comprehensive results derived from the security and performance evaluation of the proposed signcryption scheme. The findings are discussed in terms of achieved security properties and computational efficiency, drawing comparisons with several existing protocols.

14.8.1 Security Analysis

The security capabilities of the PQLS-EVote are benchmarked against prominent existing protocols (Yu et al. 2022; Guo and Deng 2020; Cai et al. 2020; Yu and Wang 2021, 2023). Table 14.4 provides a side-by-side comparison of critical features such as unforgeability, confidentiality, full-key anonymity, threshold signcryption, post-quantum resistance, decentralization, and efficiency.

It is evident from the comparison that while some existing protocols support a few of these features, none of them offer a holistic security solution. For instance, schemes like Guo and Deng (2020) and Cai et al. (2020) lack resilience against quantum adversaries and decentralized execution. Protocols like Yu and Wang (2021, 2023) incorporate threshold signcryption but fall short in other crucial areas, such as efficiency and decentralization.

In contrast, the PQLS-EVote ensures comprehensive protection by supporting all the essential security features. Notably, it is the only scheme to offer full-key anonymity, threshold signcryption, quantum resilience, and decentralization while maintaining a high level of efficiency. These results underscore the robustness and practicality of the proposed solution in applications requiring secure, decentralized, and quantum-resistant communication, such as electronic voting systems.

14.8.2 Computation Time Evaluation

In addition to security, computational performance is a critical factor in evaluating cryptographic protocols. Table 14.5 provides the measured computational time for each scheme, detailing the signcryption time, unsigncryption time, and the total execution time.

The PQLS-EVote demonstrates significant improvements in computational efficiency. With a total execution time of approximately 1,365.49 *ms*, it outperforms all other schemes under evaluation. The next best-performing protocol (Yu and Wang 2021) exhibits a total time of 1,550.22 *ms*, while others, such as Yu and Wang (2023), incur much higher overheads exceeding 2,800 *ms*.

These results are attributed to the optimized use of lightweight cryptographic operations such as matrix-vector multiplication and Gaussian sampling, in contrast to more time-intensive operations like bilinear pairings and scalar multiplications utilized by some other schemes. The careful design of the protocol, with a balanced trade-off between security and efficiency, enables the proposed solution to achieve better computational performance without compromising on security guarantees.

14.8.3 Summary of Findings

The evaluation results reveal that our PQLS-EVote provides a strong combination of security features and computational efficiency. Unlike existing methods that compromise on quantum resistance, decentralization, or operational speed, the proposed protocol achieves all of these objectives simultaneously. Therefore, it represents a practical and forward-looking approach to securing sensitive applications, particularly in decentralized and quantum-vulnerable environments such as e-voting systems.

14.9 CONCLUSION AND FUTURE SCOPE

In this chapter, a robust and quantum-resilient signcryption scheme is presented grounded in lattice-based cryptographic constructs, specifically tailored for the domain of electronic

voting. Our approach addresses the growing threat posed by quantum computing to conventional cryptographic systems by leveraging the computational hardness of lattice problems. To safeguard voter anonymity and ballot confidentiality, the scheme integrates a dynamic key generation mechanism that ensures unique keys are used for every signcryption operation, thereby reinforcing user privacy. Furthermore, our work introduces an advanced threshold-based mechanism using the (t,n) secret-sharing approach. This technique partitions critical voting information into multiple parts, making it accessible only when a minimum threshold of authorized participants collaborates. This not only strengthens resistance against insider attacks but also ensures data availability and integrity in a distributed environment. The formal security analysis, carried out under the ROM, confirms that the proposed framework offers strong protection against forgery, tampering, and privacy breaches. Additionally, the comparative performance evaluation highlights the computational viability of our system, demonstrating superior efficiency over existing lattice and classical counterparts in the context of secure electronic voting. Overall, the contribution lays a foundational step toward developing trustworthy, scalable, and quantum secure voting infrastructures. It underscores the potential of lattice-based signcryption combined with threshold cryptography to meet the evolving demands of future digital election systems.

In the near future, it is planned to extend the proposed methodology by integrating blockchain technology to enhance auditability and transparency in electronic voting. Further research can also explore optimizing the scheme's performance in large-scale elections, incorporating post-quantum secure identity management systems, and implementing the solution in real-world pilot projects to evaluate its practical deployment challenges and user experience. Additionally, investigating hybrid approaches that combine lattice-based schemes with other quantum-resistant primitives may further enhance the security and adaptability of digital voting frameworks.

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